10. **n**th ROOTS OF UNITY:

If \(1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}\) are the \(n\), \(n\)th root of unity then:

(i) They are in G.P. with common ratio \(e^{2\pi i/n}\)

(ii) \(1 + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0\) if \(p\) is not an integral multiple of \(n\)

(iii) \((1 - \alpha_1)(1 - \alpha_2)\ldots(1 - \alpha_{n-1}) = n\) and \((1 + \alpha_1)(1 + \alpha_2)\ldots(1 + \alpha_{n-1}) = 0\) if \(n\) is even and 1 if \(n\) is odd.

(iv) \(1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} = 1\) or \(-1\) according as \(n\) is odd or even.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED:

(i) \(\cos \theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos \left(\frac{n+1}{2}\right)\theta\).

(ii) \(\sin \theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin \left(\frac{n+1}{2}\right)\theta\).

Note: If \(\theta = (2\pi/n)\) then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS:

(A) If \(z_1\) & \(z_2\) are two complex numbers then the complex number \(z = \frac{nz_1 + mz_2}{m + n}\) divides the joins of \(z_1\) & \(z_2\) in the ratio \(m:n\).

Note: (i) If \(a, b, c\) are three real numbers such that \(az_1 + bz_2 + cz_3 = 0\) where \(a + b + c = 0\) and \(a, b, c\) are not all simultaneously zero, then the complex numbers \(z_1, z_2\) & \(z_3\) are collinear.

(ii) If the vertices \(A, B, C\) of a \(\Delta\) represent the complex nos. \(z_1, z_2, z_3\) respectively, then:

(a) Centroid of the \(\Delta ABC = \frac{z_1 + z_2 + z_3}{3}\)

(b) Orthocentre of the \(\Delta ABC = \frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}\) OR \(z_1 \tan A + z_2 \tan B + z_3 \tan C = 0\)

(c) Incentre of the \(\Delta ABC = (az_1 + bz_2 + cz_3) / (a + b + c)\)

(d) Circumcentre of the \(\Delta ABC = \frac{(z_1 \sin A + z_2 \sin B + z_3 \sin C)}{(\sin 2A + \sin 2B + \sin 2C)}\).

(B) \(\text{amp}(z) = \theta\) is a ray emanating from the origin inclined at an angle \(\theta\) to the \(x\)-axis.

(C) \(|z - a| = |z - b|\) is the perpendicular bisector of the line joining \(a\) to \(b\).

(D) The equation of a line joining \(z_1\) & \(z_2\) is given by:

\[z = z_1 + t(z_2 - z_1)\] where \(t\) is a parameter.

(E) \(z = z_2 + (1 + it)\) where \(t\) is a real parameter is a line through the point \(z_2\) and perpendicular to \(oz_1\).

(F) The equation of a line passing through \(z_1\) & \(z_2\) can be expressed in the determinant form as

\[
\begin{vmatrix}
z & z & 1 \\
z_1 & z_1 & 1 \\
z_2 & z_2 & 1 \\
\end{vmatrix} = 0.
\]

This is also the condition for three complex numbers to be collinear.

(G) Complex equation of a straight line through two given points \(z_1\) & \(z_2\) can be written as \(z(z_1 - z_2) - z_1(z_1 - z_2) + (z_2 - z_1)z = 0\), which on manipulating takes the form as \(z\bar{z} + \alpha z + \alpha\bar{z} + r = 0\) where \(r\) is real and \(\alpha\) is a non zero complex constant.

(H) The equation of circle having centre \(z_0\) & radius \(\rho\) is:

\[|z - z_0| = \rho\]

or \(z\bar{z} + z_0z + \bar{z}_0z - \rho^2 = 0\) which is of the form \(z\bar{z} + \alpha z + \alpha\bar{z} + r = 0\) where \(r\) is real and \(\alpha\) is a complex constant, \(\sqrt{\alpha \bar{\alpha} - r}\).

Circle will be real if \(\alpha \bar{\alpha} - r \geq 0\).

(I) The equation of the circle described on the line segment joining \(z_1\) & \(z_2\) as diameter is:

\[(i) \arg(z - z_2) = \pm \frac{\pi}{2} \quad \text{or} \quad (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0\]

(J) Condition for four given points \(z_1, z_2, z_3, z_4\) to be concyclic is, the number
13. (a) **Reflection points for a straight line**:  
Two given points $P$ & $Q$ are the reflection points for a given straight line if the given line is the right bisector of the segment $PQ$. Note that the two points denoted by the complex numbers $z_1$ & $z_2$ will be the reflection points for the straight line $\overline{C}z + \alpha \overline{C}z + r = 0$ if and only if $\overline{C}z_1 + \alpha \overline{C}z_2 + r = 0$. Where $r$ is real and $\alpha$ is non-zero complex constant.

(b) **Inverse points w.r.t. a circle**:  
Two points $P$ & $Q$ are to be inverse w.r.t. a circle with centre $O$ and radius $r$, if:

(i) The point $O$, $P$, $Q$ are collinear and on the same side of $O$.  
(ii) $OP \cdot OQ = r^2$.

Note that the two points $z_1$ & $z_2$ will be the inverse points w.r.t. the circle $zz + \alpha \overline{z}z + r = 0$ if and only if $z_1z_2 + \alpha \overline{z}_1z_2 + r = 0$.

14. **PTOLEMY'S THEOREM**: It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides. i.e.,

\[
|z_1 - z_2| |z_3 - z_4| = |z_1 - z_4| + |z_2 - z_4|.
\]

15. **LOGARITHM OF A COMPLEX QUANTITY**:

(i) \[\log_e(\alpha + i \beta) = \frac{1}{2} \log_e(\alpha^2 + \beta^2) + i \left(2\pi + \tan^{-1}\frac{\beta}{\alpha}\right)\text{ where } n \in \mathbb{I}.
\]

(ii) \[\alpha^i\] represents a set of positive real numbers given by $e^{-\frac{2\pi n}{3}}$, $n \in \mathbb{I}.$

---

**EXAMPLE EXERCISE**

Q.1 Simplify and express the result in the form of $a + bi$.

(a) \[(1 + 2i)^2\] \hspace{1cm} (b) \[-i(9 + 6i)(2 - i)^{-1}\] \hspace{1cm} (c) \[(4i^3 - 1)^2\] \hspace{1cm} (d) \[\frac{3 + 2i}{2 - 5i} + \frac{3 - 2i}{2 + 5i}\]

Q.2 Given that $x, y \in \mathbb{R}$, solve:

(a) $(x + 2y) + i(2x - 3y) = 5 - 4i$  
(b) $(x + iy)^3 = 9 + 4i$

(c) $x^2 - y^2 = (2x + y)i$  
(d) $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$

(e) $4x^2 + 3xy + (2xy - 3x^2)y = 4y^2(2x^2) + (3xy - 2y)^2i$

Q.3 Find the square root of:

(a) $9 + 40i$  
(b) $-1 - 60i$  
(c) $50i$

Q.4 Among the complex numbers $z$ satisfying the condition $|z + 3 - \sqrt{3}i| = \sqrt{3}$, find the number having the least positive argument.

Q.5 Solve the following equations over C and express the result in the form $a + ib$, $a, b \in \mathbb{R}$.

(a) $ix^2 - 3x - 2i = 0$  
(b) $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$

Q.6 Locate the points representing the complex number $z$ on the Argand plane:

(a) $|z + 1 - 2i| = \sqrt{7}$  
(b) $|z - 4| + |z + 1| = 4$  
(c) $\frac{z - 3}{z + 3} = 3$  
(d) $|z - 3| = |z - 6|$ (e) $\frac{2 + i}{4i + (1 + i)^2}$

Q.7 If $a$ and $b$ are real numbers between $0\&1$ such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3$ form an equilateral triangle, then find the values of $a'$ and $b'$.

Q.8 For what real values of $x$ & $y$ are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex?

Q.9 Find the modulus, argument and the principal argument of the complex numbers.

(i) $6(\cos 310^\circ - i \sin 310^\circ)$ \hspace{1cm} (ii) $-2(\cos 30^\circ + i \sin 30^\circ)$ \hspace{1cm} (iii) \[-\frac{2 + i}{4i + (1 + i)^2}\]

Q.10 If $(x + iy)^{1/3} = a + bi$; prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

Q.12.2 If $a + ib = p + qi$, prove that $p^2 + q^2 = \frac{a^2 + b^2}{c^2 + d^2}$.

(b) Let $z_1, z_2, z_3$ be the complex numbers such that $z_1 + z_2 + z_3 = z_1z_2 + z_2z_3 + z_3z_1 = 0$. Prove that $|z_1| = |z_2| = |z_3|$.

Q.13 Let $z$ be a complex number such that $z \in \mathbb{C}\text{ and } \frac{1 + z + z^2}{1 - z + z^2} \in \mathbb{R}$, then prove that $|z| = 1$.

Q.14 Prove the identity, $|1 - z|z_2| - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$.
For any two complex numbers, prove that \( |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 \left[ |z_1|^2 + |z_2|^2 \right] \). Also give the geometrical interpretation of this identity.

Find all non-zero complex numbers \( Z \) satisfying \( Z = iZ^2 \).

If the complex numbers \( z_1, z_2, \ldots, z_n \) lie on the unit circle \( |z| = 1 \) then show that
\[
|z_1 + z_2 + \ldots + z_n| = |z_1 - z_2 + \ldots + z_n| = |z_1^2 + z_2^2 + \ldots + z_n^2|.
\]

Find the modulus, argument and the principal argument of the complex numbers.

The value of the expression,
\[
(1 + x)^n + (1 - x)^n = 2 \cos \frac{nx}{4}.
\]

Show that the sum
\[
\sum_{k=1}^{2n} \sin \frac{2\pi k}{2n+1} \cos \frac{2\pi k}{2n+1}
\]
simplifies to a pure imaginary number.

If \( x = \cos \theta + i \sin \theta \) \& \( 1 + \sqrt{1 - a^2} = na \), prove that \( 1 + a \cos \theta = \frac{a}{2n} (1 + nx)(1 + \frac{n}{\sqrt{a}}) \).

The number \( t \) is real and not an integral multiple of \( \pi/2 \). The complex numbers \( x_1 \) and \( x_2 \) are the roots of the equation,
\[
\tan^2(t) \cdot x^2 + \tan(t) \cdot x + 1 = 0
\]
Show that \( (x_1)^n + (x_2)^n = 2 \cos \frac{2\pi n}{3} \cot^n(t) \).

Simplify and express the result in the form of \( a + bi \):

\( -i (9 + 6i) (2 - i)^{-1} \)

\( \left( \frac{4i^3 - i}{2i + 1} \right)^2 \)

\( \frac{2 + i}{2 - i} - \frac{2 - i}{2 + i} \)

\( \sqrt{1 + \sqrt{-1}} \)

Find the modulus, argument and the principal argument of the complex numbers.

\( i = 1 + \cos \left( \frac{10\pi}{9} \right) + i \sin \left( \frac{10\pi}{9} \right) \)

\( (\tan 1 - i)^2 \)

\( \frac{i - 1}{i \left( 1 - \cos \frac{2\pi}{5} \right) + \sin \frac{2\pi}{5}} \)

Given that \( x, y \in \mathbb{R} \), solve:

\( (x + 2y) + i (2x - 3y) = 5 - 4i \)

\( \frac{x}{1 + 2i} + \frac{y}{3 + 2i} = \frac{5 + 6i}{8} \)

\( x^2 - y^2 - i (2x + y) = 2i \)

\( \frac{1 + 2i}{(2 + 3i)x^2 - (3 - 2i)y} = 2x - 3y + 5i \)

\( 4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i \)

Let \( Z \) is complex satisfying the equation, \( z^2 - (3 + i)z + m + 2i = 0 \), where \( m \in \mathbb{R} \).
Suppose the equation has a real root, then find the value of \( m \).

(b) \( a, b, c \) are real numbers in the polynomial, \( P(Z) = 2Z^3 + aZ^2 + bZ + c \).

If two roots of the equation \( P(Z) = 0 \) are 2 and \( i \), then find the value of \( a' \).

Q.5(a) Find the real values of \( x \) & \( y \) for which \( z_1 = 9y^2 - 4 - 10ix \) and \( z_2 = 8y^2 - 20i \) are conjugate complex of each other.

(b) Find the value of \( x^4 - x^3 + x^2 + 3x - 5 \) if \( x = 2 + 3i \).

Q.7(a) If \( iZ^2 + Z - Z + i = 0 \), then show that |\( Z \)| = 1.

(b) Let \( z_1 \) and \( z_2 \) be two complex numbers such that \( \frac{|z_1 - 2z_2|}{2 - z_1z_2} = 1 \) and \( |z_2| \neq 1 \), find \( |z_1| \).

(c) Let \( z_1 = 10 + 6i \) & \( z_2 = 4 + 6i \). If \( z \) is any complex number such that the argument of \( \frac{z - z_1}{z - z_2} \) is \( \frac{\pi}{4} \), then prove that \( |z - 7 - 9i| = 3\sqrt{2} \).

Q.8 Show that the product,

\[
\left[ 1 + \frac{1+i}{2} \right] \left[ 1 + \frac{(1+i)^2}{2} \right] \left[ 1 + \frac{(1+i)^2}{2} \right] \cdots \left[ 1 + \frac{(1+i)^{2n}}{2} \right]
\]

is equal to \( \left( 1 - \frac{1}{2^n} \right)^{1+i} \) where \( n \geq 2 \).

Q.9 Let \( a \& b \) be complex numbers (which may be real) and let,

\( Z = z^2 + (a + b + 3i)z^2 + (ab + 3ia + 2ib - 2)z + 2abi - 2a \).

(i) Show that \( Z \) is divisible by \( z + b + i \). (ii) Find all complex numbers \( z \) for which \( Z = 0 \).

(iii) Find all purely imaginary numbers \( a \& b \) when \( z = 1 + i \) and \( Z \) is a real number.

Q.10 Interpret the following loci in \( z \in C \).

(a) \( 1 < |z - 2i| < 3 \) (b) \( \text{Re} \left( \frac{z + 2i}{iz + 2} \right) \leq 4 \) (\( z \neq 2i \))

(c) \( \text{Arg} (z + i) - \text{Arg} (z - i) = \frac{\pi}{2} \) (d) \( \text{Arg} (z - a) = \frac{\pi}{3} \) where \( a = 3 + 4i \).

Q.11 Prove that the complex numbers \( z_1 \) and \( z_2 \) and the origin form an isosceles triangle with vertical angle \( \frac{2\pi}{3} \) if \( z_1^2 + z_2^2 + z_1z_2 = 0 \).

Q.12 \( P \) is a point on the Argand diagram. On the circle with OP as diameter two points Q & R are taken such that \( \angle P0Q = \angle Q0R = \theta \). If \( \theta \) is the origin & P, Q & R are represented by the complex numbers \( Z_1, Z_2 \) & \( Z_3 \) respectively, show that \( Z_2^2 + \cos \theta Z_3 \).

Q.13 Let \( z_1, z_2, z_3 \) are three pair wise distinct complex numbers and \( t_1, t_2, t_3 \) are non-negative real numbers such that \( t_1 + t_2 + t_3 = 1 \). Prove that the complex number \( z = t_1z_1 + t_2z_2 + t_3z_3 \) lies inside a triangle with vertices \( z_1, z_2, z_3 \) or on its boundary.

Q.14 If \( a \text{cis} \alpha, b \text{cis} \beta, c \text{cis} \gamma \) represent three distinct collinear points in an Argand's plane, then prove the following:

(i) \( \sum ab \sin (\alpha - \beta) = 0 \).

(ii) \( (a \text{cis} \alpha) \sqrt{b^2 + c^2 - 2bc \cos (\beta - \gamma)} + (b \text{cis} \beta) \sqrt{a^2 + c^2 - 2ac \cos (\alpha - \gamma)} \)

\( \mp (c \text{cis} \gamma) \sqrt{a^2 + b^2 - 2ab \cos (\alpha - \beta)} = 0 \).

Q.15 Find all real values of the parameter \( a \) for which the equation \( (a - 1)z^4 - 4z^2 + a + 2 = 0 \) has only pure imaginary roots.

Q.16 Let \( A \equiv z_1, B \equiv z_2, C \equiv z_3 \) are three complex numbers denoting the vertices of an acute angled triangle. If the origin ‘O’ is the orthocentre of the triangle, then prove that

\( z_1 \bar{z}_2 + \bar{z}_1z_2 = z_2 \bar{z}_3 + z_3 \bar{z}_2 = z_3 \bar{z}_1 + \bar{z}_3z_1 \).

Q.17 If the complex number \( P(w) \) lies on the standard unit circle in an Argand’s plane and \( z = (aw + b)(w - c)^{-1} \) then, find the locus of \( z \) and interpret it. Given a, b, c are real.

Q.18(a) Without expanding the determinant at any stage, find \( K \in R \) such that

\[
\begin{bmatrix} 4i & 8i & 4 + 3i \\ -8 + i & 16i & i \\ 4 + Ki & i & 8i \end{bmatrix}
\]

has purely imaginary value.

(b) If A, B and C are the angles of a triangle
Q.19 If \( w \) is an imaginary cube root of unity then prove that:
(a) \((1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8) \ldots \) to 2n factors = \( 2^n \).
(b) If \( w \) is a complex cube root of unity, find the value of
\[(1 + w)(1 + w^2)(1 + w^4)(1 + w^8) \ldots \) to n factors.

Q.20 Prove that
\[
\frac{1 + \sin \theta + i \cos \theta}{1 + \sin \theta - i \cos \theta} = \cos \left( \frac{\pi}{2} - n\theta \right) + i \sin \left( \frac{\pi}{2} - n\theta \right).
\]
Hence deduce that
\[
\left( 1 + \sin \frac{\pi}{5} + i \cos \frac{\pi}{5} \right)^5 + i \left( 1 + \sin \frac{\pi}{5} - i \cos \frac{\pi}{5} \right)^5 = 0.
\]

Q.21 If \( \cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2 \) then prove that:
(a) \( \Sigma \cos 2\alpha = 0 = \Sigma \sin 2\alpha \)
(b) \( \Sigma \sin(\alpha + \beta) = 0 = \Sigma \cos(\alpha + \beta) \)
(c) \( \Sigma \sin^2 \alpha = \Sigma \cos^2 \alpha = 3/2 \)
(d) \( \Sigma \sin 3\alpha = 3 \sin(\alpha + \beta + \gamma) \)
(e) \( \Sigma \cos 3\alpha = 3 \cos(\alpha + \beta + \gamma) \)
(f) \( \cos^3(\alpha + \beta) + \cos^3(\beta + \gamma) + \cos^3(\gamma + \alpha) = 3 \cos(\alpha + \beta) \cos(\beta + \gamma) \cos(\gamma + \alpha) \) where \( \theta \in \mathbb{R} \).

Q.22 Resolve \( Z^5 + 1 \) into linear and quadratic factors with real coefficients. Deduce that:
\[
4 \sin \frac{\pi}{10} \cdot \cos \frac{\pi}{5} = 1.
\]

Q.23 If \( x = 1 + i\sqrt{3} \); \( y = 1 - i\sqrt{3} \) & \( z = 2 \), then prove that \( x^n + y^n = z^n \) for every prime \( p \geq 3 \).

Q.24 If the expression \( z^5 - 32 \) can be factorised into linear and quadratic factors over real coefficients as \( (z^2 - 2)(z^2 - 4)(z^2 - 8) \), then find the value of \( (p^2 + 2p) \).

Q.25(a) Let \( z = x + iy \) be a complex number, where \( x \) and \( y \) are real numbers. Let \( A \) and \( B \) be the sets defined by \( A = \{ z \mid |z| \leq 2 \} \) and \( B = \{ z \mid (1 - i)z + (1 + i)z \geq 4 \} \). Find the area of the region \( A \cap B \).

(b) For all real numbers \( x \), let the mapping \( f(x) = \frac{1}{x - i} \), where \( i = \sqrt{-1} \). If there exist real number \( a, b, c \) and \( d \) for which \( f(a), f(b), f(c) \) and \( f(d) \) form a square on the complex plane. Find the area of the square.

**EXERCISE-2**

Q.1 If \( p, q, r \) are the moduli of non-zero complex numbers \( u, v, w \) respectively,
\[
\begin{align*}
p &= |u|, \\
q &= |v|, \\
r &= |w|,
\end{align*}
\]
prove that, \( \arg \frac{w}{v} = \arg \left( \frac{w - u}{v - u} \right) \).

Q.2 The equation \( x^3 = 9 + 46i \) where \( i = \sqrt{-1} \) has a solution of the form \( a + bi \) where \( a \) and \( b \) are integers. Find the value of \( (a^3 + b^3) \).

Q.3 Show that the locus formed by \( z \) in the equation \( z^3 + iz = 1 \) never crosses the co-ordinate axes in the Argand’s plane. Further show that \( |z| = \sqrt{\frac{-\text{Im}(z)}{2 \text{Re}(z) \text{Im}(z) + 1}} \).

Q.4 If \( \omega \) is the fifth root of 2 and \( x = \omega + \omega^2 \), prove that \( x^5 = 10x^2 + 10x + 6 \).

Q.5 Prove that, with regard to the quadratic equation \( z^2 + (p + ip')z + q + iq' = 0 \) where \( p, p', q, q' \) are all real.
(i) if the equation has one real root then \( q'^2 - pp' + qp'^2 = 0 \).
(ii) if the equation has two equal roots then \( p^2 - p^2 = 4q + pp' = q'^2 \).
State whether these equal roots are real or complex.

Q.6 If the equation \( (z + 1)^2 + z = 0 \) has roots \( z_1, z_2, \ldots, z_7 \), find the value of
\[
\sum_{r=1}^{7} \text{Re}(Z_r).
\]

Q.7 Find the roots of the equation \( Z^7 = (Z + 1)^8 \) and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation
\[
\left( \frac{2 \sin \frac{m\pi}{n} }{n} \right)^2 Z^2 + \left( \frac{2 \sin \frac{m\pi}{n} }{n} \right)^2 Z + 1 = 0.
\]

Q.8 Dividing \( f(z) \) by \( z - i \), we get the remainder \( i \) and dividing it by \( z + i \), we get the remainder
Let \( z \) be any two arbitrary complex numbers then prove that:
\[
|z_1 + z_2| \geq \frac{1}{2}(|z_1| + |z_2|)|\overline{z_1} + \overline{z_2}|
\]

If \( Z_r, r = 1, 2, 3, \ldots, 2m \), \( m \in \mathbb{N} \) are the roots of the equation
\[
Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots + Z + 1 = 0
\]
then prove that \( \sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m \)

If \((1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \) \( (n \in \mathbb{N}) \), prove that:

(a) \( C_0 + C_4 + C_8 + \ldots = \frac{1}{2} \left[ 2^{n-1} + 2^{n/2} \cos \frac{n\pi}{4} \right] \)
(b) \( C_1 + C_5 + C_9 + \ldots = \frac{1}{2} \left[ 2^{n-1} + 2^{n/2} \sin \frac{n\pi}{4} \right] \)
(c) \( C_2 + C_6 + C_{10} + \ldots = \frac{1}{2} \left[ 2^{n-1} - 2^{n/2} \cos \frac{n\pi}{4} \right] \)
(d) \( C_3 + C_7 + C_{11} + \ldots = \frac{1}{2} \left[ 2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4} \right] \)
(e) \( C_0 + C_3 + C_6 + C_9 + \ldots = \frac{1}{3} \left[ 2^n + 2 \cos \frac{n\pi}{3} \right] \)

Let \( z_1, z_2, z_3, z_4 \) be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that:

(i) \( 2z_2 = (1 + i) z_1 + (1 - i) z_3 \)
(ii) \( 2z_4 = (1 - i) z_1 + (1 + i) z_3 \)

Show that all the roots of the equation
\[
\frac{1 + i x}{1 - i x} = \frac{1 + i a}{1 - i a}
\]
are real and distinct.

Prove that:

(a) \( \cos x + \binom{n}{1} \cos 2x + \binom{n}{2} \cos 3x + \ldots + \binom{n}{n} \cos (n+1)x = 2^n \cos^2 \frac{x}{2} \cdot \cos \left( \frac{n+2}{2} \right) x \)
(b) \( \sin x + \binom{n}{1} \sin 2x + \binom{n}{2} \sin 3x + \ldots + \binom{n}{n} \sin (n+1)x = 2^n \sin^2 \frac{x}{2} \cdot \sin \left( \frac{n+2}{2} \right) x \)
(c) \( \cos \left( \frac{2\pi}{2n+1} \right) + \cos \left( \frac{4\pi}{2n+1} \right) + \cos \left( \frac{6\pi}{2n+1} \right) + \ldots + \cos \left( \frac{2n\pi}{2n+1} \right) = -\frac{1}{2} \) when \( n \in \mathbb{N} \).

Show that all roots of the equation \( a_1 z^n + a_2 z^{n-1} + \ldots + a_n = n \) where \( |a_i| \leq 1, i = 0, 1, 2, \ldots, n \) lie outside the circle with centre at the origin and radius \( \frac{n-1}{n} \).

The points A, B, C of the triangle ABC are each equal to \( \frac{1}{2}(\pi - \alpha) \). Show that
\[
(z_2 - z_3)^2 = 4 \left( z_3 - z_1 \right) \left( z_1 - z_2 \right) \sin^2 \frac{\alpha}{2}
\]

Show that the equation \( \frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + \ldots + \frac{A_n^2}{x - a_n} = k \) has no imaginary root, given that:
\( a_1, a_2, a_3, \ldots, a_n, A_1, A_2, A_3, \ldots, A_n, k \) are all real numbers.

Let a, b, c be distinct complex numbers such that \( \frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k \). Find the value of k.

Let \( \alpha, \beta \) be fixed complex numbers and \( z \) is a variable complex number such that
\[
|z - \alpha|^2 + |z - \beta|^2 = k.
\]
Find out the limits for 'k' such that the locus of \( z \) is a circle. Find also the centre and radius of the circle.

C is the complex number. \( f: C \rightarrow R \) is defined by \( f(z) = |z^3 - z + 2| \). What is the maximum value of \( f \) on the unit circle \( |z| = 1 \)?

Let \( f(x) = \log_{\cos 3x} (\cos 2x) \) if \( x \neq 0 \) and \( f(0) = K \) (where \( i = \sqrt{-1} \)) is continuous at \( x = 0 \) then find the value of K. Use of L'Hospital's rule or series expansion not allowed.

If \( z_1, z_2 \) are the roots of the equation \( az^2 + bz + c = 0 \), with \( a, b, c > 0 \); \( 2b^2 > 4ac > b^2 \); \( z_1 \in \text{third quadrant} \); \( z_2 \in \text{second quadrant} \) in the argand's plane then, show that
Q.23 Find the set of points on the argand plane for which the real part of the complex number
\( (1 + i) z^2 \) is positive where \( z = x + iy \), \( x, y \in \mathbb{R} \) and \( i = \sqrt{-1} \).

Q.24 If \( a \) and \( b \) are positive integer such that \( N = (a + ib)^3 - 107i \) is a positive integer. Find \( N \).

Q.25 If the biquadratic \( x^4 + ax^3 + bx^2 + cx + d = 0 \) \((a, b, c, d \in \mathbb{R})\) has 4 non real roots, two with sum \( 3 + 4i \) and the other two with product \( 13 + i \). Find the value of \( b \).

\[ \text{arg} \left( \frac{z_1}{z_2} \right) = 2 \cos^{-1} \left( \frac{b^2}{4ac} \right) \]

Q.26 Evaluate: \( \sum_{p=1}^{32} (3p + 2) \left( \sum_{q=1}^{10} \left( \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right)^p \).

Q.27 Let \( z_1 \) and \( z_2 \) be roots of the equation \( z^2 + pz + q = 0 \), where the co-efficients \( p \) and \( q \) may be complex numbers. Let \( A \) and \( B \) represent \( z_1 \) and \( z_2 \) in the complex plane. If \( \angle AOB = \alpha \neq 0 \) and \( OA = OB \), where \( O \) is the origin. Prove that \( p^2 = 4q \cos^2 \left( \frac{\alpha}{2} \right) \).

Q.28 If \( i = 3 \) and \( z \) is a positive integer, find the equation whose roots are, \( 128 \) and \( 0 \).

Q.29 If \( \alpha = e^{\frac{2\pi i}{3}} \) and \( f(x) = a_0 + \sum_{k=1}^{\infty} a_k x^k \), then find the value of, \( f(x) + f(\alpha x) + \ldots + f(\alpha^k x) \) independent of \( \alpha \).

Q.30 If \( z_1, z_2, z_3 \) are complex numbers such that \( |z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1 \), then
\[ |z_1 + z_2 + z_3| \text{ is:} \]
(A) equal to 1 \hspace{5cm} (B) less than 1 \hspace{5cm} (C) greater than 3 \hspace{5cm} (D) equal to 3

Q.31 Given, \( z = \cos \frac{2\pi}{2n + 1} + i \sin \frac{2\pi}{2n + 1} \), \( n \) a positive integer, find the equation whose roots are, \( \alpha = z + z^{2} + \ldots + z^{2n-1} \) and \( \beta = z^{2} + z^{3} + \ldots + z^{2n} \).

Q.32 If \( \arg (z) < 0 \), then \( \arg (-z) - \arg (z) = \)
(A) \( \pi \) \hspace{5cm} (B) \( -\pi \) \hspace{5cm} (C) \( -\frac{\pi}{2} \) \hspace{5cm} (D) \( \frac{\pi}{2} \)

Q.33 The complex numbers \( z_1, z_2 \) and \( z_3 \) satisfying \( \frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2} \) are the vertices of a triangle which is
(A) of area zero \hspace{5cm} (B) right-angled isosceles \hspace{5cm} (C) equilateral \hspace{5cm} (D) obtuse – angled isosceles
Q.10 Find all those roots of the equation \( z^{12} - 56z^6 - 512 = 0 \) whose imaginary part is positive.

JEE 2000, 3 out of 100

Q.11(a) Let \( z_1, z_2 \) be \( n \)th roots of unity which subtend a right angle at the origin. Then \( n \) must be of the form
(A) \( 4k + 1 \)  
(B) \( 4k + 2 \)  
(C) \( 4k + 3 \)  
(D) \( 4k \)

JEE 2001 (Scr) 1 + 1 out of 35

(b) Let \( z \) and \( z' \) be \( n \)th roots of unity which subtend a right angle at the origin. Then \( n \) must be of the form
(A) \( 4k + 1 \)  
(B) \( 4k + 2 \)  
(C) \( 4k + 3 \)  
(D) \( 4k \)

JEE 2002 (Scr) 3+3

(c) Let \( \alpha, \beta \neq 1 \) be a root of the equation
\[ z^p + z^q = 0 \] where \( p, q \) are distinct primes.

Show that either \( 1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0 \) or \( 1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0 \), but not both together.

JEE 2002, (5)

Q.12(a) If \( z_1 \) and \( z_2 \) are two complex numbers such that \( |z_1| < 1 < |z_2| \) then prove that
\[ \frac{1}{z_1} < \frac{1}{z_2} \]

(b) Prove that there exists no complex number \( z \) such that \( |z| < \frac{1}{3} \) and \( \sum_{r=1}^{n} a_r z^r = 1 \) where \( |a_r| < 2 \).

Q.13(a) \( \omega \) is an imaginary cube root of unity. If \((1 + \omega^m)^m = (1 + \omega^n)^n \), then least positive integral value of \( m \) is
(A) 6  
(B) 5  
(C) 4  
(D) 3

JEE 2004 (Scr)

(b) Find centre and radius of the circle determined by all complex numbers \( z = x + iy \) satisfying
\[ (z - \alpha)(z - \beta) = k, \]
where \( \alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2 \) are fixed complex and \( k \neq 1 \).

Q.14(a) The locus of \( z \) which lies in shaded region is best represented by
(A) \( z : |z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4} \)  
(B) \( z : |z - i| < 2, |\arg(z - i)| < \frac{\pi}{2} \)  
(C) \( z : |z - 1| < 2, |\arg(z - 1)| < \frac{\pi}{2} \)  
(D) \( z : |z| = 1, z \neq 1 \)

Q.15 If \( w = \alpha + i\beta \) where \( \beta \neq 0 \) and \( z \neq 1 \), satisfies the condition that \( \frac{w - Wz}{1 - z} \) is purely real, then the set of values of \( z \) is
(A) \( \{ z : |z| = 1 \} \)  
(B) \( \{ z : z = \overline{z} \} \)  
(C) \( \{ z : z \neq 1 \} \)  
(D) \( \{ z : |z| = 1, z \neq 1 \} \)

JEE 2006, 3

ANSWER KEY

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.1</td>
<td>( \frac{7}{25}, \frac{24}{25}, \frac{21}{5}, \frac{12}{5}, \frac{8}{9}, \frac{22}{5} )</td>
</tr>
<tr>
<td>Q.2</td>
<td>( x = 1, y = 2; ) (b) ( 2, 9; ) (c) ( -2, 2; ) (d) ( 1, 1; ) (e) ( \frac{3K}{2}, K \in R )</td>
</tr>
</tbody>
</table>
Q.3 \( f(z) \) is maximum when \( z = K + 1 \), Arg \( z = \frac{\pi}{2} \).

Q.4 (a) \(-160\); (b) \(-77 + 108i\)

Q.6 (a) \(-i, -2i\); (b) \(\frac{3-5i}{2}\) or \(-1+i/2\)

Q.7 (a) on a circle of radius \(\sqrt{7}\) with centre \((-1, 2)\); (b) on a unit circle with centre at origin (c) on a circle with centre \((-15/4, 0)\) and radius 9/4; (d) a straight line

Q.8 \(a = b = 2 - \sqrt{3}\)

Q.9 \(x = 1, y = -4\) or \(x = -1, y = -4\)

Q.10 (i) Modulus = 6, Arg = \(2k \pi + \frac{5\pi}{18}\) \((K \in I)\), Principal Arg = \(\frac{5\pi}{18}\) \((K \in I)\)

(ii) Modulus = 2, Arg = \(2k \pi + \frac{7\pi}{6}\) \((K \in I)\), Principal Arg = \(-\frac{5\pi}{6}\)

(iii) Modulus = \(\sqrt{5}\), Arg = \(2k \pi - \tan^{-1}2\) \((K \in I)\), Principal Arg = \(-\tan^{-1}2\)

Q.16 (a) \(\frac{\sqrt{3}}{2} - \frac{i}{2}, \frac{-\sqrt{3}}{2} + \frac{i}{2}\); (b) \(\frac{n(n+1)}{2}\) \(\pm n\)

Q.17 \(\frac{x^2}{64} + \frac{y^2}{48} = 1\)

Q.18 (c) 64

Q.21 A

Q.22 (a) \((1, 1)\); (b) \(\left|\frac{n(n+1)}{2}\right|\) - \(n\)

**EXERCISE-1**

Q.1 (a) \(\frac{21}{5} - \frac{12}{5}i\); (b) \(3 + 4i\); (c) \(-\frac{8}{29} + 0i\); (d) \(\frac{22}{5}i\); (e) \(\pm \sqrt{2} + 0i\) or \(0 \pm \sqrt{2}i\)

Q.2 (i) Principal Arg \(z = -\frac{4\pi}{9}\); \(|z| = 2 \cos 4\pi/9\);

(ii) Principal Arg \(z = 2n \pi + (2 - \pi)\). Principal Arg = \(2n \pi + (2 - \pi)\)

(iii) Principal Arg \(z = \frac{n \pi}{2}\) and \(|z| = \frac{\sqrt{2}}{3}\)

(iv) Principal Arg \(z = \frac{n \pi}{2}\) and \(|z| = \frac{2}{3}\)

Q.3 (a) \(x = 1, y = 2\); (b) \(x = 1 \& y = 2\); (c) \((-2, 2)\) or \((-\frac{2}{3}, \frac{2}{3})\); (d) \((1, 1)\); (e) \(x = K, y = \frac{3K}{2}\) \(K \in R\)

Q.4 (a) 2; (b) \(-11/2\)

Q.5 (a) \((-2, -2)\); (b) \((-77 + 108i)\)

Q.6 (a) \(z = (2 + i)\) or \((1 - 3i)\); (b) \(z = \frac{3 + 4i}{4}\)

Q.7 (b) \(2\)

Q.9 (i) \(z = (b + i)\); \(-2i, a\)

(ii) \(z = \left(-\frac{2ti}{3t + 5}, ti\right)\) where \(t \in R - \left\{\frac{-5}{3}\right\}\)

Q.10 (a) The region between the co-centric circles with centre at \((0, 2)\) and radii 1 & 3 units

(b) region outside or on the circle with centre \(\frac{1}{2} + 2i\) and radius \(\frac{1}{2}\).

(c) semi-circle (in the 1st & 4th quadrant) \(x^2 + y^2 = 1\) (d) a ray emanating from the point \((3 + 4i)\) directed away from the origin & having equation \(\sqrt{3}x - y + 4 - 3\sqrt{3} = 0\)

Q.15 \([-3, -2]\)

Q.17 \((1 - c^2) \left|z^2 - 2(a + bc) \left(Re\ z\right) - a^2 - b^2\right| = 0\)

Q.18 (a) \(K = 3\); (b) \(-4\)

Q.19 (b) one if \(n\) is even; \(-w^2\) if \(n\) is odd

Q.22 \((Z + 1)(Z - 2Z \cos 36^\circ + 1)(Z - 2Z \cos 108^\circ + 1)\)

Q.24 4

Q.25 (a) \(\pi = 2\); (b) \(1/2\)

**EXERCISE-2**

Q.23 \(35\)

Q.6 (a) \(-\frac{7}{2}\), (b) zero

Q.8 \(\frac{i z}{2} + \frac{1}{2} + i\)

Q.18 \(\omega\) or \(-\omega^2\)

Q.19 \(k > \frac{1}{2}\) \(|\alpha - \beta|^2|2\)

Q.20 \(|f(z)|\) is maximum when \(z = \omega\), where \(\omega\) is the cube root unity and \(|f(z)| = \sqrt{13}\)

Q.21 \(K = -\frac{4}{9}\)

Successful People Replace the words like; "wish", "try" & "should" with "I Will". Ineffective People don’t.
required set is constituted by the angles without their boundaries, whose sides are the straight lines

\[ y = (\sqrt{2} - 1) x \text{ and } y + (\sqrt{2} + 1) x = 0 \text{ containing the } x - axis \]

**EXERCISE-3**

<table>
<thead>
<tr>
<th>Q.1</th>
<th>48(1 - i)</th>
<th>Q.3</th>
<th>(a) D</th>
<th>(b) B</th>
</tr>
</thead>
</table>
| Q.4 | \[ Z = \frac{(29 + 20\sqrt{2}) + i(\pm 15 + 25\sqrt{2})}{82} ; \]  
\[ Z = \frac{(29 - 20\sqrt{2}) + i(\pm 15 - 25\sqrt{2})}{82} \] | Q.5 | (a) C   |
| Q.6 | 7A_0 + 7A_1x^7 + 7A_{14}x^{14} | Q.7 | (a) A   | (b) A   |
| Q.8 | \[ z^2 + z + \frac{\sin^2 n\theta}{\sin^2 \theta} = 0, \text{ where } \theta = \frac{2\pi}{2n+1} \] | Q.9 | (a) C   | (b) D   |
| Q.10 | \[ \pm 1 + i\sqrt{3} ; \sqrt{2}i \] | Q.11 | (a) B   | (b) B   |
| Q.12 | 13 \[ \text{ where } z \] | Q.13 | (a) D   | (b) Centre = \[ \frac{k^2 - 3}{k^2 - 1} \]  
\[ \text{Radius} = \frac{1}{k^2 - 1} \sqrt{\alpha - k^2\beta^2} - (k^2. \beta^2 - |\alpha|^2)(k^2 - 1) \] | Q.14 | (a) A, (b) B, (c) \[ z_2 = \sqrt{3}i ; z_3 = (1 - \sqrt{3})i ; z_4 = (1 + \sqrt{3})i \] | Q.15 | D   |

**EXERCISE-4**

**Part : (A) Only one correct option**

1. If \[ |z| = 1 \text{ and } \omega = \frac{z - 1}{z + 1} \text{ (where } z \neq -1), \text{ the Re}(\omega) \text{ is} \]

   (A) 0  
   (B) \[ \frac{1}{|z + 1|^2} \]  
   (C) \[ \frac{z}{|z + 1|^2} \]  
   (D) \[ \frac{1}{|z + 1|^2} \]  

   [IIT – 2003, 3]

2. The locus of \[ z \] which lies in shaded region (excluding the boundaries) is best represented by

   \[ \text{arg}(z) = \frac{\pi}{4} \]

   (A) \[ z : |z + 1| > 2 \text{ and } |\arg(z + 1)| < \frac{\pi}{4} \]  
   (B) \[ z : |z| > 2 \text{ and } |\arg(z - 1)| < \frac{\pi}{4} \]  
   (C) \[ z : |z| = 1 \text{ and } |\arg(z + 1)| < \frac{\pi}{4} \]  
   (D) \[ z : |z - 1| > 2 \text{ and } |\arg(z + 1)| < \frac{\pi}{4} \]  

   [IIT – 2005, 3]

3. If \[ w = \alpha + i\beta, \text{ where } \beta \neq 0 \text{ and } z \neq 1, \text{ satisfies the condition that} \]

   \[ \frac{w}{1 - z} \text{ is purely real, then the set of} \]

   (A) \[ z : |z + 1| > 2 \text{ and } |\arg(z + 1)| < \frac{\pi}{4} \]  
   (B) \[ z : |z - 1| > 2 \text{ and } |\arg(z - 1)| < \frac{\pi}{4} \]  
   (C) \[ z : |z - 1| > 2 \text{ and } |\arg(z + 1)| < \frac{\pi}{4} \]  
   (D) \[ z : |z| > 2 \text{ and } |\arg(z - 1)| < \frac{\pi}{4} \]  

   [IIT – 2006, (3, –1)]

4. If \[ (\sqrt{3} + i)^{100} = (a + ib) \text{ (a + ib)}, \text{ then } b \text{ is equal to} \]

   (A) \[ \sqrt{3} \]  
   (B) \[ \sqrt{2} \]  
   (C) \[ 1 \]  
   (D) none of these

5. If \[ \text{Re} \left( \frac{z - 8i}{z + 6} \right) = 0, \text{ then } z \text{ lies on the curve} \]

   (A) \[ x^2 + y^2 + 6x - 8y = 0 \]  
   (B) \[ 4x - 3y + 24 = 0 \]  
   (C) \[ 4a + b \]  
   (D) none of these

6. If \[ n_1, n_2 \text{ are positive integers then} \]

   (A) \[ (1 + i)^{n_1} + (1 - i)^{n_2} \]  
   (B) \[ (1 + i)^{n_1} + (1 - i)^{n_2} \]  
   (C) \[ (1 + i)^{n_1} + (1 - i)^{n_2} \]  
   (D) \[ n_1, n_2 \text{ are any two positive integers} \]

   The three vertices of a triangle are represented by the complex numbers, \( 0, z_1 \) and \( z_2 \). If the triangle is equilateral, then

   (A) \[ z_1^2 - z_2^2 = z_1z_2 \]  
   (B) \[ z_2^2 - z_1^2 = z_1z_2 \]  
   (C) \[ z_1^2 + z_2^2 = z_1z_2 \]  
   (D) \[ z_1^2 + z_2^2 + z_1z_2 = 0 \]

7. If \[ x^2 - x + 1 = 0 \text{ then the value of} \]

   \[ \sum_{n=1}^{5} \left( \frac{x^n + 1}{x^n} \right)^2 \]

   (A) 8  
   (B) 10  
   (C) 12  
   (D) none of these

Successful People Replace the words like; "wish", "try" & "should" with "I Will". Ineffective People don't.
9. If \( \alpha \) is nonreal and \( \alpha = \frac{\sqrt{3}}{2} \) then the value of \( 2^{1 + n + \frac{\alpha^2}{2} - \alpha^2} \) is equal to
(A) 4  
(B) 2  
(C) 1  
(D) none of these

10. If \( z = x + iy \) and \( z^{13} = a - ib \) then \( \frac{x}{a} - \frac{y}{b} = k (a^2 - b^2) \) where \( k = \)
(A) 1  
(B) 2  
(C) 3  
(D) 4

11. \[ \left[ \frac{-1 + i\sqrt{3}}{2} \right]^6 + \left[ \frac{-1 - i\sqrt{3}}{2} \right]^6 + \left[ \frac{-1 + i\sqrt{3}}{2} \right]^5 + \left[ \frac{-1 - i\sqrt{3}}{2} \right]^5 \] is equal to :
(A) 1  
(B) \(-1\)  
(C) 2  
(D) none

12. Expressed in the form \( r (\cos \theta + i \sin \theta) \), \(-2 + 2i\) becomes :
(A) \( 2\sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right) \)  
(B) \( 2\sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right) \)  
(C) \( 2\sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right) \)  
(D) \( \sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right) \)

13. The number of solutions of the equation in \( z, z \bar{z} - (3 + i) z - (3 - i) \bar{z} - 6 = 0 \) is :
(A) 0  
(B) 1  
(C) 2  
(D) infinite

14. If \( |z| = \max \{|z-1|, |z+1|\} \) then
(A) \( |z + \bar{z}| = \frac{1}{2} \)  
(B) \( z + \bar{z} = 1 \)  
(C) \( |z + \bar{z}| = 1 \)  
(D) none of these

15. If \( P, P' \) represent the complex number \( z_1 \) and its additive inverse respectively then the complex equation of the circle with \( PP' \) as a diameter is
(A) \( \frac{z}{\overline{z}_1} = \frac{\bar{z}_1}{z} \)  
(B) \( z \bar{z} + z_1 \bar{z}_1 = 0 \)  
(C) \( z \bar{z} + \bar{z} z_1 = 0 \)  
(D) none of these

16. The points \( z_1 = 3 + \sqrt{3} i \) and \( z_2 = 2 \sqrt{3} + 6 i \) are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors \( z_1 \) and \( z_2 \) is :
(A) \( z = \frac{(3 + 2\sqrt{3}) \sqrt{3} + 2}{2} i \)  
(B) \( z = 5 + 5i \)  
(C) \( z = -1 - i \)  
(D) none

17. The expression \( \frac{1 + i \tan \alpha}{1 - i \tan \alpha} \) when simplified reduces to :
(A) zero  
(B) \( 2 \sin \alpha \)  
(C) \( 2 \cos \alpha \)  
(D) none

18. All roots of the equation, \( (1 + z)^6 + z^2 = 0 \) :
(A) lie on a unit circle with centre at the origin  
(B) lie on a unit circle with centre at \((-1, 0)\)  
(C) lie on the vertices of a regular polygon with centre at the origin  
(D) are collinear

19. Points \( z_1, z_2 \) are adjacent vertices of a regular octagon. The vertex \( z_3 \) adjacent to \( z_2 (z_3 \neq z_1) \) is represented by :
(A) \( z_2 + \frac{1}{\sqrt{2}} (1 \pm i) (z_1 + z_2) \)  
(B) \( z_2 + \frac{1}{\sqrt{2}} (1 \pm i) (z_1 - z_2) \)  
(C) \( z_2 + \frac{1}{\sqrt{2}} (1 \pm i) (z_2 - z_1) \)  
(D) none of these

20. If \( z = x + iy \) then the equation of a straight line \( Ax + By + C = 0 \) where \( A, B, C \in R \) can be written on the complex plane in the form \( \overline{a} z + a \overline{z} + 2C = 0 \) where \( \overline{a} \) is equal to :
(A) \( \frac{A + iB}{2} \)  
(B) \( \frac{A - iB}{2} \)  
(C) \( A + iB \)  
(D) none

21. The points of intersection of the two curves \(|z - 3| = 2\) and \(|z| = 2\) in an argand plane are:
(A) \( \frac{1}{2} \left( 7 \pm i \sqrt{3} \right) \)  
(B) \( \frac{1}{2} \left( 3 \pm i \sqrt{7} \right) \)  
(C) \( \frac{3}{2} \pm i \sqrt{\frac{7}{2}} \)  
(D) \( \frac{7}{2} \pm i \sqrt{\frac{3}{2}} \)

22. The equation of the radical axis of the two circles represented by the equations, \(|z + 2| = 3\) and \(|z - 2 - 3i| = 4\) on the complex plane is :
(A) \( 3iz - 3i \bar{z} + 2 = 0 \)  
(B) \( 3iz - 3i \bar{z} + 2 = 0 \)  
(C) \( iz - i \bar{z} + 1 = 0 \)  
(D) \( 2iz - 2i \bar{z} + 3 = 0 \)

23. If \( \prod_{p=1}^{n} e^{ip\theta} = 1 \) where \( \Pi \) denotes the continued product, then the most general value of \( \theta \) is :
(A) \( \frac{2n\pi}{r(r-1)} \)  
(B) \( \frac{2n\pi}{r(r+1)} \)  
(C) \( \frac{4n\pi}{r(r-1)} \)  
(D) \( \frac{4n\pi}{r(r+1)} \)

24. The set of values of \( a \in R \) for which \( x^4 + a(x-1) x + 5 = 0 \) will have a pair of conjugate imaginary roots is
(A) \( R \)  
(B) \( \{1\} \)  
(C) \(|a|^2 - 2a + 21 > 0\)  
(D) none of these

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### Exercise-5

1. Given that \(x, y \in \mathbb{R}\), solve: \(4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i\)

2. If \(\alpha\) & \(\beta\) are any two complex numbers, prove that:
   \[
   \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|.
   \]

3. If \(a, b\) are the numbers between 0 and 1, such that the points \(z_1 = a + i, z_2 = 1 + \beta i\) and \(z_3 = 0\) form an equilateral triangle, then find \(\alpha\) and \(\beta\).

4. ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers \(1 + i\) and \(2 - i\) respectively, then find the complex number corresponding to A.

5. Show that the sum of the \(p^n\) powers of \(n^n\) roots of unity:
   (a) is zero, when \(p\) is not a multiple of \(n\).
   (b) is equal to \(n\), when \(p\) is a multiple of \(n\).

6. If \((1 + x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots\), then prove that:
   \[
   \begin{align*}
   p_0 - p_2 + p_4 - \ldots & = 2^{n/2} \cos \frac{n \pi}{4} \\
   p_1 - p_3 + p_5 - \ldots & = 2^{n/2} \sin \frac{n \pi}{4}
   \end{align*}
   \]

7. Prove that, \(\log_b \left( \frac{1}{1 - e^{i \theta}} \right) = \log_b \left( \frac{1}{2 \cos \frac{\theta}{2}} \right) + i \left( \frac{\pi - \theta}{2} \right)\)

8. If \(i^j \to \infty = A + iB\), principal values only being considered, prove that:
   \[
   \begin{align*}
   \tan \frac{\pi}{2} & = B \\
   A^2 + B^2 & = e^{-nB}
   \end{align*}
   \]

9. Prove that the roots of the equation, \((x - 1)^n = x^n\) are \(\frac{1}{2} \left( 1 + i \cot \frac{\pi}{n} \right)\), where \(r = 0, 1, 2, \ldots \) \((n - 1)\) & \(n \in \mathbb{N}\).

10. If \(\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2\) then prove that:
   \[
   \begin{align*}
   \sum \cos 2\alpha & = 0 = \sum \sin 2\alpha \\
   \sum \sin (\alpha + \beta) & = 0 = \sum \cos (\alpha + \beta) \\
   \sum \sin 3\alpha & = 3 \sin (\alpha + \beta + \gamma) \\
   \sum \cos 3\alpha & = 3 \cos (\alpha + \beta + \gamma) \\
   \sum \sin^2 \alpha & = \sum \cos^2 \alpha = 3/2 \\
   \cos^3 (\theta + \alpha) + \cos^3 (\theta + \beta) + \cos^3 (\theta + \gamma) & = 3 \cos (\theta + \alpha) \cos (\theta + \beta) \cos (\theta + \gamma)
   \end{align*}
   \]

\(\theta \in \mathbb{R}\).
11. If \( \alpha, \beta, \gamma \) are roots of \( x^3 - 3x^2 + 3x + 7 = 0 \) (and \( \omega \) is imaginary cube root of unity), then find the value of
\[
\frac{\alpha - 1}{\beta + 1} + \frac{\beta - 1}{\gamma + 1} + \frac{\gamma - 1}{\alpha + 1}.
\]

12. Given that \( |z - 1| = 1 \), where \( z \) is a point on the argand plane. Show that \( \frac{z - 2}{z} = i \tan (\arg z) \).

13. P is a point on the Argand diagram. On the circle with OP as diameter two points Q & R are taken such that \( \angle POQ = \angle QOR = \beta \). If \( O \) is the origin & P, Q & R are represented by the complex numbers \( z_1, z_2, z_3 \) respectively, show that : \( z_2^2 \cos 2\theta = z_1 \), \( z_3 \cos^2 \theta \).

14. Find an expression for \( \tan 7\theta \) in terms of \( \tan \theta \), using complex numbers. By considering \( \tan 7\theta = 0 \), show that \( x = \tan^2 (3\pi/7) \) satisfies the cubic equation \( x^3 - 21x^2 + 35x - 7 = 0 \).

15. If \( (1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n \) (\( n \in \mathbb{N} \)), prove that : \( C_2 + C_6 + C_{10} + \ldots = \frac{1}{2} \left[ 2^{n-1} - 2^{n/2} \cos \frac{n\pi}{4} \right] \).

16. Prove that : \( \cos \left( \frac{2\pi}{2n+1} \right) + \cos \left( \frac{4\pi}{2n+1} \right) + \cos \left( \frac{6\pi}{2n+1} \right) + \ldots + \cos \left( \frac{2n\pi}{2n+1} \right) = -\frac{1}{2} \) when \( n \in \mathbb{N} \).

17. Show that all the roots of the equation \( a_1z^n + a_2z^{n-1} + a_3z^{n-2} + \ldots + a_n = 0 \) are roots of unity, then prove that \( a_1, a_2, a_3, \ldots, a_n \) & \( A_1, A_2, A_3, \ldots, A_n, k \) are all real numbers.

18. Let \( z_1, z_2, z_3 \) be three distinct complex numbers satisfying, \( 1/2z_1 - 1/2 = 1/2z_2 - 1/2 = 1/2z_3 - 1/2 \). Let \( A, B \) & \( C \) be the points represented in the Argand plane corresponding to \( z_1, z_2 \) & \( z_3 \) respectively. Prove that \( z_1 + z_2 + z_3 = 3 \) if and only if \( D \) is a point inside the circle with centre \( O \) and radius \( 2/3 \).

19. Let \( \alpha, \beta \) be fixed complex numbers and \( z \) is a variable complex number such that,
\[
|z - \alpha|^2 + |z - \beta|^2 = k.
\]
Find out the limits for \( k \) such that the locus of \( z \) is a circle. Find also the centre and radius of the circle.

20. If \( 1 + \alpha_1, 1 + \alpha_2, \ldots, 1 + \alpha_n \) are the \( n \) \( n \)-th roots of unity, then prove that
\[
(1 - \alpha_1) (1 - \alpha_2) (1 - \alpha_3) \ldots (1 - \alpha_n) = n.
\]
Hence prove that \( \sin \frac{\pi}{n}, \sin \frac{2\pi}{n}, \sin \frac{3\pi}{n}, \ldots, \sin \frac{(n - 1)\pi}{n}, \frac{n}{2} \leq \sin \frac{n\pi}{2n - 1} \leq n \).

21. Find the real values of the parameter \( n \) for which at least one complex number \( z = x + iy \) satisfies both the equality \( |z - ai| = a + 4 \) and the inequality \( |z - 2| < 1 \).

22. Prove that, with regard to the quadratic equation \( z^2 + (p + ip') z + q + iq' = 0 \); where \( p, p', q, q' \) are all real.
(a) if the equation has one real root then \( q'^2 - pp' < 0 \); and \( p, p', q, q' \) are all real.
(b) if the equation has two equal roots then \( p^2 - p'^2 = 4q \) & \( pp' = 2q' \).
State whether these equal roots are real or complex.

23. The points \( A, B \) & \( C \) depict the complex numbers \( z_1, z_2, z_3 \) respectively on a complex plane & the angle \( B \) & \( C \) of the triangle \( ABC \) are each equal to \( 1/2 (\pi - \alpha) \). Show that
\[
(z_2 - z_3)^2 = 4 (z_3 - z_1) (z_1 - z_2) \sin^2 \frac{\alpha}{2}.
\]

24. If \( z_1, z_2, z_3 \) are the affixes of three points \( A, B \) & \( C \) respectively and satisfy the condition \( |z_1 - z_2| = |z_1| + |z_2| \) and \( |z_1 - z_3| = |z_1| + |z_1 - z_3| \) then prove that \( \Delta ABC \) is a right angled.

25. If \( 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be the roots of \( x^5 - 1 = 0 \), then prove that
\[
\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} = \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} = \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} = \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \omega.
\]

26. If one of the vertices of the square circumscribing the circle \( |z - 1| = \sqrt{2} \) is \( 2 + \sqrt{3} \) i. Find the other vertices of the square.

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**EXERCISE-4**

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**EXERCISE-5**

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