WAVES ON A STRING

WAVES

Wave motion is the phenomenon that can be observed almost everywhere around us, as well it appears in almost every branch of physics. Surface waves on bodies of mater are commonly observed. Sound waves and light waves are essential to our perception of the environment. All waves have a similar mathematical description, which makes the study of one kind of wave useful for the study of other kinds of waves. In this chapter, we will concentrate on string waves, which are type of a mechanical waves. Mechanical waves require a medium to travel through. Sound waves, water waves are other examples of mechanical waves. Light waves are not mechanical waves, these are electromagnetic waves which do not require medium to propagate.

Mechanical waves originate from a disturbance in the medium (such as a stone dropping in a pond) and the disturbance propagates through the medium. The forces between the atoms in the medium are responsible for the propagation of mechanical waves. Each atom exerts a force on the atoms near it, and through this force the motion of the atom is transmitted to the others. The atoms in the medium do not, however, experience any net displacement. As the wave passes, the atoms simply move back and forth. Again for simplicity, we concentrate on the study of harmonic waves (that is those that can be represented by sine and cosine functions).

TYPES OF MECHANICAL WAVES

Mechanical waves can be classified according to the physical properties of the medium, as well as in other ways.

1. Direction of particle motion :
Waves can be classified by considering the direction of motion of the particles in the medium as wave passes. If the disturbance travels in the x direction but the particles move in a direction, perpendicular to the x axis as the wave passes it is called a transverse wave. If the motion of the particles were parallel to the x axis then it is called a longitudinal wave. A wave pulse in a plucked guitar string is a transverse wave. A sound wave is a longitudinal wave.

2. Number of dimensions :
Waves can propagate in one, two, or three dimensions. A wave moving along a taut string is a one dimensional wave. A water wave created by a stone thrown in a pond is a two dimensional wave. A sound wave created by a gunshot is a three-dimensional wave

3. Periodicity :
A stone dropped into a pond creates a wave pulse, which travels outward in two dimensions. There may be more than one ripple created, but there is still only one wave pulse. If similar stones are dropped in the same place at even time intervals, then a periodic wave is created.

4. Shape of wave fronts : The ripples created by a stone dropped into a pond are circular in shape. A sound wave propagating outward from a point source has spherical wavefronts. A plane wave is a three dimensional wave with flat wave fronts.
(Far away from a point source emitting spherical waves, the waves appear to be plane waves.)

A solid can sustain transverse as well as longitudinal wave. A fluid has no well-defined form or structure to maintain and offer far more resistance to compression than to a shearing force. Consequently, only longitudinal wave can propagate through a gas or within the body of an ideal (non viscous) liquid.
However, transverse waves can exist on the surface of a liquid. In the case of ripples on a pond, the force restoring the system to equilibrium is the surface tension of the water, whereas for ocean waves, it is the force of gravity.
Also, if disturbance is restricted to propagate only in one direction and there is no loss of energy during propagation, then shape of disturbance remains unchanged.

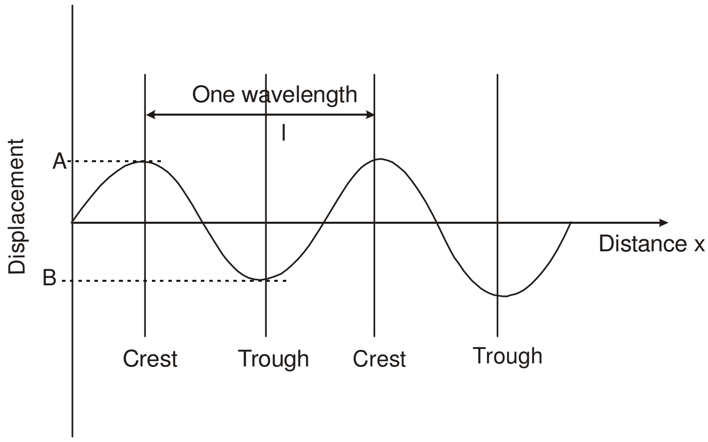
DESCRIBING WAVES :

Two kinds of graph may be drawn - displacement-distance and displacement-time.
A displacement - distance graph for a transverse mechanical wave shows the displacement y of the vibrating particles of the transmitting medium at different distance x from the source at a certain instant i.e. it is like a photograph showing shape of the wave at that particular instant.

The maximum displacement of each particle from its undisturbed position is the amplitude of the wave. In the figure 1, it is OA or OB.

The wavelength λ of a wave is generally taken as the distance between two successive crests or two successive trough. To be more specific, it is the distance between two consecutive points on the wave which have same phase.

A displacement-time graph may also be drawn for a wave motion, showing how the displacement of one particle at a particular distance from the source varies with time. If this is simple harmonic variation then the graph is a sine curve.



WAVE LENGTH, FREQUENCY, SPEED

If the source of a wave makes f vibrations per second, so too will the particles of the transmitting medium. That is, the frequency of the waves equals frequency of the source.

When the source makes one complete vibration , one wave is generated and the disturbance spreads out a distance λ from the source. If the source continues to vibrate with constant frequency f, then f waves will be produced per second and the wave advances a distance f λ in one second. If v is the wave speed then

$$v = f\lambda$$

This relationship holds for all wave motions.

Travelling wave :

Imagine a horizontal string stretched in the x direction. Its equilibrium shape is flat and straight. Let y measure the displacement of any particle of the string from its equilibrium position, perpendicular to the string. If the string is plucked on the left end, a pulse will travel to the right. The vertical displacement y of the left end of the string (x = 0) is a function of time.

i.e. $y(x = 0, t) = f(t)$

If there are no frictional losses, the pulse will travel undiminished, retaining its original shape. If the pulse travels with a speed v, the 'position' of the wave pulse is x = vt. Therefore, the displacement of the particle at point x at time t was originated at the left end at time $t - \frac{x}{v}$. [y, (x, t) is function of both x and t]. But the displacement of the left end at time t is f(t) thus at time $t - \frac{x}{v}$, it is $f(t - \frac{x}{v})$.

Therefore

$$y(x, t) = y\left(x = 0,\ t - \frac{x}{v}\right) \qquad = f\left(t - \frac{x}{v}\right)$$

This can also be expressed as

$$\Rightarrow \quad \frac{f}{v}(vt - x) \qquad\qquad \Rightarrow \quad -\frac{f}{v}(x - vt)$$

$$y(x, t) = g(x - vt)$$

using any fixed value of t (i.e. at any instant), this shows shape of the string.

If the wave is travelling in – x direction, then wave equation is written as

$$y(x, t) = f\left(t + \frac{x}{v}\right)$$

The quantity x – vt is called phase of the wave function. As phase of the pulse has fixed value

$$x - vt = \text{const.}$$

Taking the derivative w.r.t. time $\frac{dx}{dt} = v$

where v is the phase velocity although often called wave velocity. It is the velocity at which a particular phase of the disturbance travels through space.

In order for the function to represent a wave travelling at speed v, the three quantities x, v and t must appear in the combination (x + vt) or (x – vt). Thus $(x - vt)^2$ is acceptable but $x^2 - v^2t^2$ is not.

Ex.-1 A wave pulse is travelling on a string at 2 m/s. Displacement y of the particle at x = 0 at any time t is given by

$$y = \frac{2}{t^2+1}$$

Find
(i) Expression of the function y =(x, t) i.e. displacement of a particle at position x and time t.
(ii) Shape of the pulse at t = 0 and t = 1s.

Sol. (i) By replacing t by $\left(t - \frac{x}{v}\right)$, we can get the desired wave function i.e.

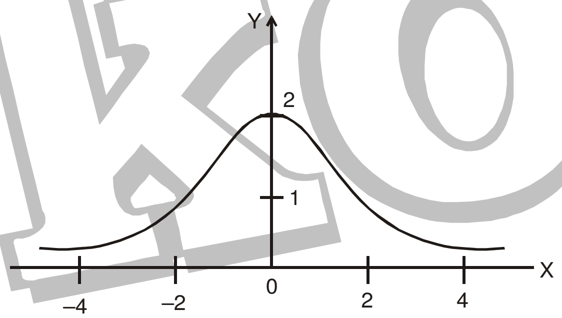
$$y = \frac{2}{\left(t - \frac{x}{2}\right)^2 + 1}$$

(ii) We can use wave function at a particular instant, say t = 0, to find shape of the wave pulse using different values of x.

at t = 0 $\quad y = \frac{2}{\frac{x^2}{4} + 1}$

at	x = 0	y = 2
	x = 2	y = 1
	x = – 2	y = 1
	x = 4	y = 0.4
	x = –4	y = 0.4

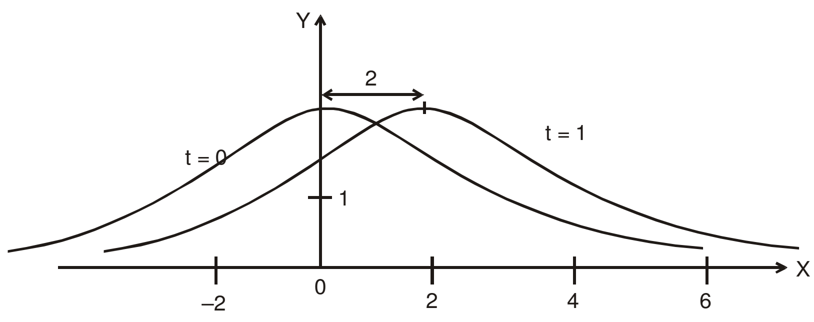
Using these value, shape is drawn.



Similarly for t = 1s, shape can be drawn. What do you conclude about direction of motion of the wave from the graphs? Also check how much the pulse has moved in 1s time interval. This is equal to wave speed. Here is the procedure :

$$y = \frac{2}{\left(1 - \frac{x}{2}\right)^2 + 1} \qquad \text{at } t = 1\text{s}$$

at	x = 2	y = 2	(maximum value)
at	x = 0	y = 1	
at	x = 4	y = 1	



The pulse has moved to the right by 2 in 2s interval.

Also as $t - \frac{x}{2}$ = constt.

Differentiating w.r.t. time

$$1 - \frac{1}{2} \cdot \frac{dx}{dt} = 0 \qquad \Rightarrow \qquad \frac{dx}{dt} = 2.$$

Ques. A wave pulse moving along the x axis is represented by the wave function

$$y(x,\ t) = \frac{2}{(x - 3t)^2 + 1}$$

where x and y are measured in cm and t is in seconds.

(i) In which direction is the wave moving?

(ii) Find speed of the wave.

(iii) Plot the waveform at t = 0, t = 2s.

Ans. (i) Positive x axis (ii) 3 cm/s.

Ques. At t = 0, a transvalues wave pulse in a wire is described by the function $y = \frac{6}{x^3 + 3}$ where x and y are in metres write the function y(x, t) that describes this wave if it is travelling in the positive x direction with a speed of 4.5 m/s.

Ans. $\frac{6}{(x - 4.5t)^2 + 3}$

TRAVELLING SINE WAVE IN ONE DIMENSION (WAVE ON STRING) :

The wave equation $y = f\left(t - \frac{x}{v}\right)$ is quite general. If holds for arbitrary wave shapes, and for transverse as well as for longitudinal waves.

A complete description of the wave requires specification of f(x). The most important case, by far, in physics and engineering is when f(x) is sinusoidal, that is, when the wave has the shape of a sine or cosine function. This is possible when the source, that is moving the left end of the string, vibrates the left end x = 0 in a simple harmonic motion. For this, the source has to continuously do work on the string and energy is continuously supplied to the string.

The equation of motion of the left end may be written as

$$f\ (t) = A \sin wt$$

where A is amplitude of the wave, that is maximum displacement of a particle in the medium from its equilibrium position w is angular frequency, that is $2\pi f$ where f is frequency of SHM of the source.

The displacement of the particle at x at time t will be

$$y = f\left(t - \frac{x}{v}\right)$$

or $$y = A \sin \omega\left(t - \frac{x}{v}\right) \qquad y = A \sin (\omega t - kx)$$

where $k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$ is called wave number. $T = \frac{2\pi}{\omega} = \frac{1}{f}$ is period of the wave, that is the time it takes to travel the distance between two adjacent crests or through (it is wavelength λ).

The wave equation $y = A \sin (\omega t - kx)$ says that at x = 0 and t = 0, y = 0. This is not necessarily the case, of source. For the same condition, y may not equal to zero. Therefore, the most general expression would involve a phase constant ϕ, which allows for other possibilities,

$$y = A \sin (\omega t - kx + \phi)$$

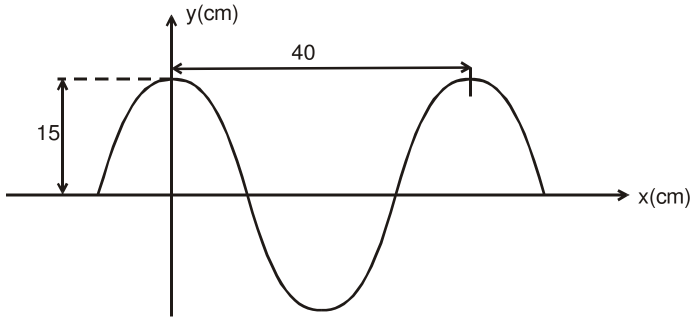
A suitable choice of ϕ allows any initial condition to be met. The term $kx - wt + \phi$ is called the phase of the wave. Two waves with the same phase (on phase differing by a multiple of 2π) are said to be "in phase". They execute the same motion at the same time.

The velocity of the particle at position x and at time t is given by

$$\frac{\partial y}{\partial t} = A\omega \cos (\omega t - kx + \phi)$$

The wave equation has been partially differentiated keeping x as constant, to specify the particle. Note that wave velocity $\frac{dx}{dt}$ is different from particle velocity while waves velocity is constant for a medium and it along the direction of string, whereas particle velocity is perpendicular to wave velocity and is dependent upon x and t.

Ex. 2 A sinusoidal wave travelling in the positive x direction has an amplitude of 15 cm, wavelength 40 cm and frequency 8 Hz. The vertical displacement of the medium at t = 0 and x = 0 is also 15 cm, as shown.



(a) Find the angular wave number, period, angular frequency and speed of the wave.

(b) Determine the phase constant ϕ, and write a general expression for the wave function.

Sol. (a) $k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{40 \text{cm}} = \frac{\pi}{20}$ rad/cm

$T = \frac{1}{f} = \frac{1}{8}$ s $\qquad \omega = 2\pi f = 16 \text{ s}^{-1}$

$v = f\lambda = 320$ cm/s

(b) It is given that A = 15 cm

and also $\quad y = 15$ cm at $x = 0$ and $t = 0$

then using $\quad y = A \sin(\omega t - kx + \phi)$

$15 = 15 \sin\phi \quad \Rightarrow \quad \sin\phi = 1$

or $\quad \phi = \frac{\pi}{2}$ rad.

Therefore, the wave function is

$$y = A \sin\left(\omega t - kx + \frac{\pi}{2}\right)$$

$$= (15 \text{ cm}) \sin\left[(16\pi s^{-1})t - \left(\frac{\pi}{20}\frac{\text{rad}}{\text{cm}}\right).x + \frac{\pi}{2}\right]$$

Ex. 3 A sinusoidal wave is travelling along a rope. The oscillator that generates the wave completes 60 vibrations in 30 s. Also, a given maximum travels 425 cm along the rope in 10.0 s. What is the wavelength?

Sol. $v = \frac{425}{10} = 42.5$ cm/s. $\qquad f = \frac{60}{30} = 2$ Hz

$\lambda = \frac{v}{f} = 21.25$ cm.

Ques. The wave function for a travelling wave on a string is given as

$$y(x, t) = (0.350 \text{ m}) \sin\left(10\pi t - 3\pi x + \frac{\pi}{4}\right)$$

(a) What are the speed and direction of travel of the wave?
(b) What is the vertical displacement of the string at t = 0, x = 0.1 m?
(c) What are wavelength and frequency of the wave?

Ans. (a) 3.33 $\hat{i}$ m/s (b) –5.48 cm (c) 0.67 m, 5 Hz.

THE LINEAR WAVE EQUATION :

By using wave function $y = A \sin(\omega t - kx + \phi)$, we can describe the motion of any point on the string. Any point on the string moves only vertically, and so its x coordinate remains constant. The transverse velocity v_y of the point and its transverse acceleration a_y are therefore

$$v_y = \left.\frac{dy}{dt}\right]_{x=\text{constant}} = \frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx + \phi) \qquad ....(1)$$

$$a_y = \left.\frac{dv_y}{dt}\right]_{x=\text{constant}} = \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(\omega t - kx + \phi) \qquad ....(2)$$

and hence

$v_{y,\,max} = \omega A$
$a_{y,\,max} = \omega^2 A$

The transverse velocity and transverse acceleration of any point on the string do not reach their maximum value simultaneously. Infact, the transverse velocity reaches its maximum value (ωA) when the displacement y = 0, whereas the transverse acceleration reaches its maximum magnitude ($\omega^2 A$) when $y = \pm A$

further

$$\left.\frac{dy}{dx}\right]_{t=\text{constant}} = \frac{\partial y}{\partial x} = -kA\cos(\omega t - kx + \phi) \qquad ....(3)$$

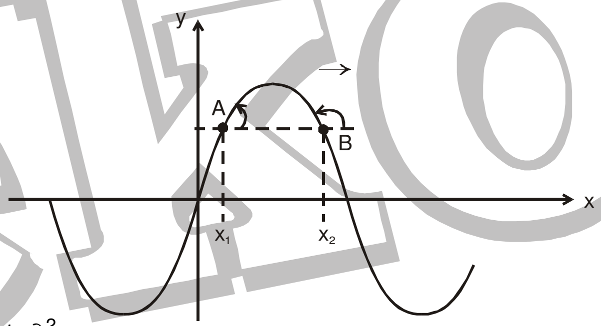
$$= \frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(\omega t - kx + \phi) \qquad ....(4)$$

From (1) and (3) $\qquad \dfrac{\partial y}{\partial t} = -\dfrac{\omega}{k}\dfrac{\partial y}{\partial x}$

$\Rightarrow \qquad v_P = -v_w \times \text{slope}$

i.e. if the slope at any point is negative, particle velocity is positive and vice-versa, for a wave moving along positive x axis i.e. v_w is positive.

For example, consider two points A and B on the y-x curve for a wave, as shown. The wave is moving along positive x-axis.



Slope at A is positive therefore at the given moment, its velocity is negative. That means it is coming downward. Reverse is the situation for particle at point B.

Now using equation (2) and (4)

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \qquad \Rightarrow \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This is known as the linear wave equation or differential equation representation of the travelling wave model. We have developed the linear wave equation from a sinusoidal mechanical wave travelling through a medium, but it is much more general. The linear wave equation successfully describes waves on strings, sound waves and also electromagnetic waves.

Ex. 4 Verify that wave function

$$y = \frac{2}{(x-3t)^2+1}$$

is a solution to the linear wave equation. x and y are in cm.

Sol. By taking partial derivatives of this function w.r.t. x and to t

$$\frac{\partial^2 y}{\partial x^2} = \frac{12(x-3t)^2-4}{[(x-3t)^2+1]^3}, \text{ and}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{108(x-3t)^2-36}{[(x-3t)^2+1]^3}$$

or $\qquad \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{9}\dfrac{\partial^2 x}{\partial t^2}$

Comparing with linear wave equation, we see that the wave function is a solution to the linear wave equation if the speed at which the pulse moves is 3 cm/s. It is apparent from wave function therefore it is a solution to the linear wave equation.

Ques. Show that the wave function $y = e^{b(x-vt)}$ is a solution of the linear wave equation.

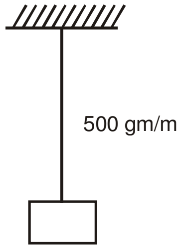
THE SPEED OF TRANSVERSE WAVES ON STRINGS

The speed of a wave on a string is given by

$$v = \sqrt{\frac{T}{\mu}}$$

where T is tension in the string (in Newtons) and μ is mass per unit length of the string (kg/m).
It should be noted that v is speed of the wave w.r.t. the medium (string).
In case the tension is not uniform in the string or string has nonuniform linear mass density then v is speed at a given point and T and μ are corresponding values at that point.

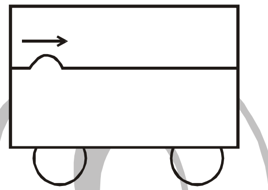
Ex. 5 Find speed of the wave generated in the string as in the situation shown. Assume that the tension is not affected by the mass of the cord.



Sol. $T = 20 \times 10 = 200$ N

$$v = \sqrt{\frac{200}{0.5}} = 20\,\text{m/s}$$

Ex. 6 A taut string having tension 100 N and linear mass density 0.25 kg/m is used inside a cart to generate a wave pulse starting at the left end, as shown. What should be the velocity of the cart so that pulse remains stationary w.r.t. ground.



Sol. Velocity of pulse = $\sqrt{\frac{T}{\mu}} = 20\,\text{m/s}$

Now $\vec{v}_{PG} = \vec{v}_{PC} + \vec{v}_{CG}$

$0 = 20\,i + \vec{v}_{CG}$

$\vec{v}_{CG} = -20i$ m/s

Ques. A uniform rope of mass m and length L hangs from a ceilling. (a) Show that the speed of a transverse wave on the rope is a funciton of y, the distance from the lower end, and is given by $v = \sqrt{gy}$. (b) Show that the time a transverse wave takes to travel the length of the rope is given by $t = 2\sqrt{L/g}$.

POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

When a travelling wave is established on a string, energy is transmitted along the direction of propagation of the wave, in form of potential energy and kinetic energy

Average Power $\langle P \rangle = 2\pi^2 f^2 A^2 \mu v$

Energy Transferred = $\int_0^t P_{av}\,dt$

Energy transfered in one time period = $P_{av}\,T$
This is also equal to the energy stored in one wavelength.

Intensity : Energy transfered per second per unit cross sectional area is called intensity of the wave.

$$I = \frac{\text{Power}}{\text{Cross sectional area}} = \frac{P}{s}$$

$$I = \frac{1}{2}\rho\omega^2 A^2 v$$

This is average intensity of the wave.
Energy density of a wave is energy per unit volume.

$$= \frac{Pdt}{svdt} = \frac{I}{v}$$

Ex. 7 Power Supplied to a Vibrating String
A string with linear mass density m = 5.00 × 10^{-2} kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?

Sol. The wave speed on the string is

$$v = \sqrt{\frac{T}{\mu}} = \left(\frac{80.0\text{N}}{5.00\times10^{-2}\,\text{kg/m}}\right)^{1/2} = \mathbf{40.0\ m/s}$$

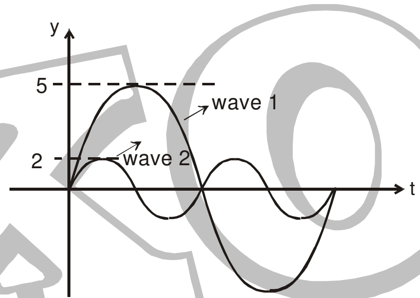
Because ϕ = 60 Hz, the angular frequency ω of the sinusoidal waves on the string has the value
$\omega = 2\pi f = 2\pi(60.0\text{ Hz}) = 377\text{ s}^{-1}$
Using these values in Equation 13.23 for the power, with A = 6.00 × 10^{-2} m,. gives

$$p = \frac{1}{2}\mu\omega^2 A^2 v$$

$$= \frac{1}{2}(5.00\times10^{-2}\text{ kg/m})\,(377\text{s}^{-1})^2\times(6.00\times10^{-2}\text{ m})^2\,(40.0\text{ m/s})$$

= 512 W.

Ex. 8 Two waves in the same medium are represented by y-t curves in the figure. Find ratio of their average intensities?



Sol. $$\frac{I_1}{I_2} = \frac{\omega_1^2 A_1^2}{\omega_2^2 A_2^2} = \frac{f_1^2\cdot A_1^2}{f_2^2\cdot A_2^2}. = \frac{1\times25}{4\times4} = \frac{25}{16}$$

Ques. A transverse wave of amplitude 0.50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m/s, what average power is the source transmitting to the wire?
Ans 49 mW

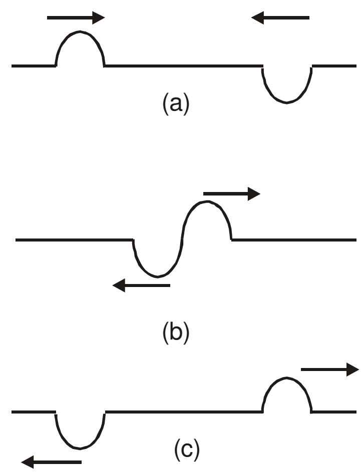
THE PRINCIPLE OF SUPERPOSITION

When two or more waves simultaneously pass through a point, the disturbance at the point is given by the sum of the disturbances each wave would produce in absence of the other wave(s).
In general, the principle of superposition is valid for small disturbances only. If the string is stretched too far, the individual displacements do not add to give the resultant displacement. Such waves are called nonlinear waves. In this course, we shall only be talking about linear waves which obey the superposition principle.
To put this rule in a mathematical from, let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that any element of the string would experience if each wave travelled alone. The displacement y(x, t) of an element of the string when the waves overlap is then given by

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

The principal of superposition can also be expressed by stating that overlapping waves algebraically add to produce a resultant wave. The principle implies that the overlapping waves do not in any way alter the travel of each other.
If we have two or more waves moving in the medium the resultant waveform is the sum of wave functions of individual waves.

Fig: A sequence of pictures showing two pulses travelling in opposite directions along a stretched string. When the two disturbances overlap they give a complicated pattern as shown in (b). In (c), they have passed each other and proceed unchanged.



An Illustrative examples of this principle is phenomena of interference and reflection of waves.

Ex. 9 Two waves passing through a region are represented by

$$y = (1.0\text{ m}) \sin [(3.14\text{ cm}^{-1})\, x - (157\text{ s}^{-1})\, t]$$

and $$y = (1.5\text{ cm}) \sin [(1.57\text{ cm}^{-1})x - (314\text{ s}^{-1})\, t].$$

Find the displacement of the particle at x = 4.5 cm at time t = 5.0 ms.

Sol. According to the principle of superposition, each wave produces its disturbance independent of the other and the resultant disturbance is equal to the vector sum of the individual disturbance. The displacements of the particle at x = 4.5 cm at time t = 5.0 ms due to the two waves are,

$$y_1 = (1.0\text{ cm}) \sin [(3.14\text{ cm}^{-1})\,(4.5\text{ cm})] - (157\text{ s}^{-1})\,(5.0 \times 10^{-3}\text{ s})]$$

$$= (1.0\text{ cm}) \sin \left[4.5\pi - \frac{\pi}{4}\right]$$

$$= (1.0\text{ cm}) \sin \left[4\pi + \frac{\pi}{4}\right] = \frac{1.0\text{ cm}}{\sqrt{2}}$$

and

$$y_2 = (1.5\text{ cm}) \sin [(1.57\text{ cm}^{-1})(4.5\text{ cm}) - (314\text{ s}^{-1})\,(5.0 \times 10^{-3}\text{ s})]$$

$$= (1.5\text{ cm}) \sin \left[2.25\pi - \frac{\pi}{2}\right] = (1.5\text{ cm}) \sin \left[2\pi - \frac{\pi}{4}\right]$$

$$= (1.5\text{ cm}) \sin \frac{\pi}{4} = -\frac{1.5\text{ cm}}{\sqrt{2}}$$

The net displacement is

$$y = y_1 + y_2 = \frac{-0.5\text{ cm}}{\sqrt{2}} = -\,0.35\text{ cm}.$$

INTERFERENCE OF WAVES GOING IN SAME DIRECTION

Suppose two identical sources send sinusoidal waves of same angular frequency ω in positive x-direction. Also, the wave velocity and hence, the wave number k is same for the two waves. One source may be situated at different points. The two waves arriving at a point then differ in phase. Let the amplitudes of the two waves be A_1 and A_2 and the two waves differ in phase by an angle ϕ. TheIr equations may be written as

$$y_1 = A_1 \sin (kx - \omega t)$$

and $$y_2 = A_2 \sin (kx - \omega t + \phi).$$

According to the principle of superposition, the resultant wave is represented by

$$y = y_1 + y_2 = A_1 \sin (kx - \omega t) + A_2 \sin (kx - \omega t + \phi).$$

we get $y = A \sin (kx - \omega t + \alpha)$

where, $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$ (A is amplitude of the resultant wave)

Also, $\tan \alpha = \dfrac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$ (α is phase difference of the resultant wave with the first wave)

Constructive and Destructive Interference

Constructive Interference :

When resultant amplitude A is maximum

$A = A_1 + A_2$

when $\cos \phi = +1$ or $\phi = 2n\pi$

where n is an integer.

Destructive interference :

When resultant amplitude A is minimum

or $A = |A_1 - A_2|$

When $\cos \phi = -1$ or $\phi = (2n + 1)\pi$

where n is an integer.

Ex. 10 Two sinusoidal waves of the same frequency travel in the same direction along a string. If $A_1 = 3.0$ cm, $A_2 = 4.0$ cm, $\phi_1 = 0$, and $\phi_2 = \pi/2$ rad, what is the amplitude of the resultant wave?

Sol. Resultant amplitude = $\sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \times \cos 90º}$

= 5 cm.

Ques. Two sinusoidal waves of the same frequency are to be sent in the same direction along a taut string. One wave has an amplitude of 5.0 mm, the other 8.0 mm. (a) What phase difference ϕ_1 between the two waves results in the smallest amplitude of the resultant wave? (b) What is that smallest amplitude? (c) What phase difference ϕ_2 results in the largest amplitude of the resultant wave? (d) What is that largest amplitude? (e) What is the resultant amplitude if the phase angle is $(\phi_1 - \phi_2)/2$?

Ans. (a) π rad; (b) 3.0 mm; (c) 0 rad; (d) 13 mm; (e) 9.4 mm

REFLECTION AND TRANSMISSION OF WAVES

A travelling wave, at a rigid or denser boundry, is reflected with a phse reversal but the reflection at an open boundary (rarer medium) takes place without any phase change. The transmitted wave is never inverted, but propagation constant k is changed.

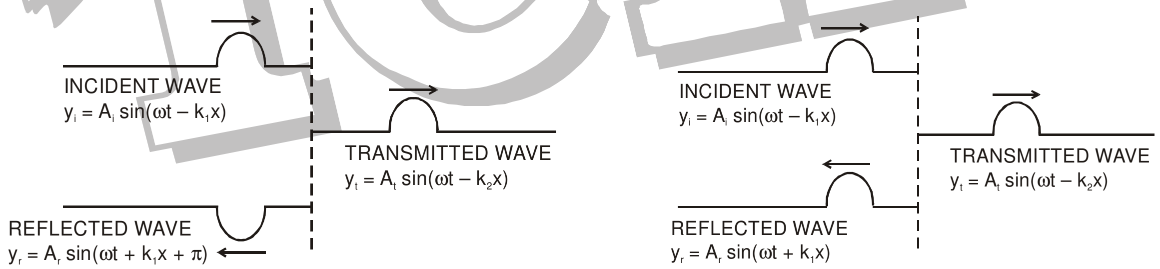


Fig. : Reflection at denser boundary

Fig. : Reflection at rarer boundary

Amplitude of reflected and transmitted waves :

v_1 and v_2 are speeds of the wave in incidenting and reflecting mediums respectively then

$A_r = \dfrac{v_2 - v_1}{v_1 + v_2} A_i$ $A_t = \dfrac{2v_2}{v_1 + v_2} . A_i$

A_r is positive if $v_2 > v_1$, i.e., wave is reflected from a rarer medium.

Ex. 11 A harmonic wave is travelling on string 1. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and that the boundary between the two strings is at x = 0. If the expression for the incident wave is,

$y_i = A_i \cos (k_1 x - \omega_1 t)$

What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?

Sol. Since $v = \sqrt{T/\mu}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

we have, $v_2 = \frac{v_1}{2}$ ... (i)

The frequency does not change, that is,

$\omega_1 = \omega_2$ ....(ii)

Also, because $k = \omega/v$, the wave numbers of the harmonic waves in the two strings are related by,

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\frac{\omega_1}{v_1} = 2k_1 \quad ....(iii)$$

The amplitudes are,

$$A_t = \left(\frac{2v_2}{v_1+v_2}\right) A_u = \left[\frac{2(v_1/2)}{v_1+(v_1/2)}\right] A_i = \frac{2}{3} A_i \quad ....(iv)$$

and
$$A_r = \left(\frac{v_2-v_1}{v_1+v_2}\right) A_u = \left[\frac{(v_1/2)-v_1}{v_1+(v_1/2)}\right] A_i = \frac{A_i}{3} \quad ....(v)$$

Now with equation (ii), (iii) and (iv), the transmitted wave can be written as,

$$y_t = \frac{2}{3} A_i \cos(2k_1 x - \omega_1 t) \qquad \textbf{Ans.}$$

Similarly the reflected wave can be expressed as,

$$= \frac{A_i}{3} \cos(k_1 x + \omega_1 t + \pi) \qquad \textbf{Ans.}$$

STANDING WAVES :

Suppose two sine waves of equal amplitude and frequency propagate on a long string in opposite directions. The equations of the two waves are given by

$y_1 = A \sin(\omega t - kx)$

and $y_2 = A \sin(\omega t + kx + \phi)$.

These waves interfere to produce what we call standing waves. To understand these waves, let us discuss the special case when $\phi = 0$.

The resultant displacements of the particles of the string are given by the principle of superposition as

$$y = y_1 + y_2$$
$$= A[\sin(\omega t - kx) + \sin(\omega t + kx)]$$
$$= 2A \sin \omega t \cos kx$$

or, $y = (2A \cos kx) \sin \omega t.$ ....

This is the required result and from this it is clear that :

1. As this equation satisfies the wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

it represents a wave. However, as it is not of the form $f(ax \pm bt)$, the wave is not travelling and so is called standing or stationary wave.

2. The amplitude of the wave

$A_s = 2A \cos kx$

is not constant but varies periodically with position (and not with time as in beats).

3. The points for which amplitude is minimum are called nodes and for these

$$\cos kx = 0, \quad \text{i.e.,} \quad kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

i.e., $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ...$ $\left[\text{as } k = \frac{2\pi}{\lambda}\right]$

i.e., in a stationary wave, nodes are equally spaced.

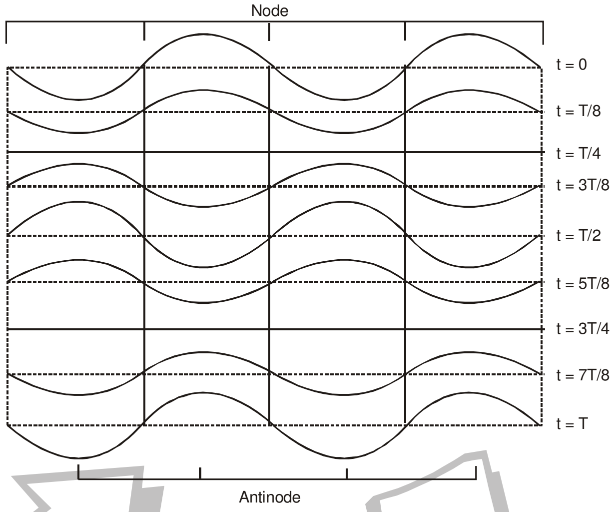
4. The points for which amplitude is maximum are called antinodes and for these,

$\cos kx = \pm 1$, i.e., $kx = 0, \pi, 2\pi, 3\pi, ...$

i.e., $x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, ...$ $\left[\text{as } k = \frac{2\pi}{\lambda}\right]$

i.e., like nodes, antinodes are also equally spaced with spacing ($\lambda/2$) and $A_{max} = \pm 2A$. Furthermore, nodes and antinodes are alternate with spacing ($\lambda/4$).

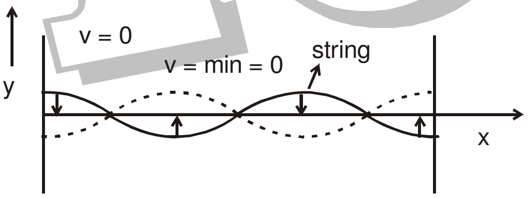
5. The nodes divide the medium into segments (or loops). All the particles in a segment vibrate in same phase, but in opposite phase with the particles in the adjacent segment. Twice in one period all the particles pass through their mean position simultaneously with maximum velocity ($A_s\omega$), the direction of motion being reversed after each half cycle.



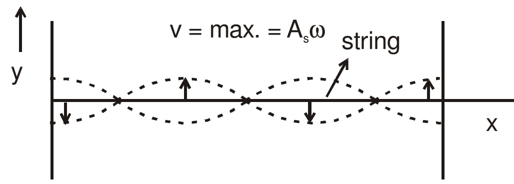
(a)

6. Standing waves can be transverse or longitudinal, e.g., in strings (under tension) if reflected wave exists, the waves are transverse-stationary, while in organ pipes waves are longitudinal-stationary.

7. As in stationary waves nodes are permanently at rest, so no energy can be transmitted across them, i.e., energy of one region (segment) is confined in that region. However, this energy oscillates between elastic potential energy and kinetic energy of the particles of the medium. When all the particles are at their extreme positions KE is minimum while elastic PE is maximum (as shown in figure A), and when all the particles (simultaneously) pass through their mean position KE will be maximum while elastic PE minimum (Figure B). The total energy confined in a segment (elastic PE + KE), always remains the same.



Elastic PE = max = E
Kinetic energy = min = 0
(A)



Elastic PE = min = 0
Kinetic energy = max = E
(B)

Ex. 12 Two waves travelling in opposite directions produce a standing wave. The individual wave functions are

$$y_1 = (4.0 \text{ cm}) \sin(3.0x - 2.0t)$$
$$y_2 = (4.0 \text{ cm}) \sin(3.0x + 2.0t)$$

where x and y are in centimeter.
(a) Find the maximum displacement of a particle of the medium at x = 2.3 cm.
(b) Find the position of the nodes and antinodes.

Sol. (a) When the two waves are summed, the rasult is a standing wave whose mathematical representaion is given by Equation, with A = 4.0 cm and k = 3.0 rad/cm;

$$y = (2A \sin kx) \cos \omega t = [(8.0 \text{ cm}) \sin 3.0\, x] \cos 2.0\, t$$

Thus, the maximum displacement of a particle at the position x = 2.3 cm is

$$y_{max} = [(8.0 \text{ cm}) \sin 3.0x]_{x = 2.3 \text{ cm}}$$
$$= (8.0 \text{ m}) \sin (6.9 \text{ rad}) = 4.6 \text{ cm}$$

(b) Because $k = 2\pi/\lambda = 3.0$ rad/cm, we see that $\lambda = 2\pi/3$cm. Therefore, the antinodes are located at

$$x = n\left(\frac{\pi}{6.0}\right) \text{ cm} \qquad (n = 1, 3, 5, ...)$$

and the nodes are located at

$$x = n\frac{\lambda}{2}\left(\frac{\pi}{3.0}\right) \text{ cm} \qquad (n = 1, 2, 3, ...)$$

Ex. 13 Two travelling waves of equal amplitudes and equal frequencies move in opposite direction along a string. They interfere to produce a standing wave having the equation.

$$y = A \cos kx \sin \omega t$$

in which A = 1.0 mm, $k = 1.57\ \text{cm}^{-1}$ and $\omega = 78.5\ \text{s}^{-1}$. (a) Find the velocity and amplitude of the component travelling waves. (b) Find the node closest to the origin in the region x > 0. (c) Find the antinode closest to the origin in the region x > 0. (d) Find the amplitude of the particle at x = 2.33 cm.

Sol. (a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2}\sin(\omega t - kx) \quad \text{and} \quad y_2 = \frac{A}{2}\sin(\omega t + kx).$$

The wave velocity (magnitude) of either of the waves is

$$v = \frac{\omega}{k} = \frac{78.5\,\text{s}^{-1}}{1.57\,\text{cm}^{-1}} = 50 \text{ cm/s; Amplitude} = 0.5 \text{ mm}.$$

(b) For a node, cos kx = 0.

The smallest positive x satisfying this relation is given by

$$kx = \pi/2$$

or, $$x = \frac{\pi}{2k} = \frac{3.14}{2\times 1.57\ \text{cm}^{-1}} = 1 \text{ cm}$$

(c) For an antinode, |cos kx| = 1.

The smallest positive x satisfying this relation is given by

$$kx = \pi$$

or, $$x = \frac{\pi}{k} = 2 \text{ cm}$$

(d) The amplitude of vibration of the particle at x is given by | A cos kx |. For the given point,

$$kx = (1.57\ \text{cm}^{-1})(2.33\ \text{cm}) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}.$$

Thus, the amplitude will be

$$(1.0 \text{ mm})\,|\cos(\pi + \pi/6)| = \frac{\sqrt{3}}{3} \text{ mm} = 0.86 \text{ mm}.$$

Ques. A string fixed at both ens is 8.40 m long and has a mass of 0.120 kg. It is subgjected to a tension of 96.0 N and set oscillating. (a) What is the speed of the waves on the string? (b) What is the lognest possible wavelength for a standing wave? (c) Give the frequency of the wave.

Ans. (a) 82.0 m/s, (b) 16.8 m, (c) 4.88 Hz.

VIBRATION OF STRING :

(a) Fixed at both ends :

Suppose a string of length L is kept fixed at the ends x = 0 and x = L. In such a system suppose we send a continuous sinusoidal wave of a certain frequency, say, toward the right. When the wave reaches the right end. It gets reflected and begins to travel back. The left-going wave then overlaps the wave, which is still travelling to the right. When the left-going wave reaches the left end, it gets reflected again and the newly reflected wave begins to travel to the right. overlapping the left-going wave. This process will continue and, therefore, very soon we have many overlapping waves, which interfere with one another. In such a system, at any point x and at any time t, there are always two waves, one moving to the left and another to the right. We, therefore, have

$y_1(x, t) = y_m \sin(kx - \omega t)$ (wave travelling in the positive direction of x-axis)

and $y_2(x, t) = y_m \sin(kx + \omega t)$ (wave travelling in the negative direction of x-axis).

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The principle of superposition gives, for the combined wave

$y'(x, t) = y_1(x, t) + y_2(x, t)$

$= y_m \sin(kx - wt) + y_m \sin(kx + \omega t)$

$= (2y_m \sin kx) \cos \omega t$

It is seen that the points of maximum or minimum amplitude stay at one position.

Nodes : The amplitude is zero for values of kx that give sin kx = 0 i.e. for,

$kx = n\pi$, for n = 0, 1, 2, 3,.....

Substituting $k = 2\pi/\lambda$ in this equation, we get

$$x = n\frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3,.....$$

The positions of zero amplitude are called the **nodes.** Note that a distanc of $\frac{\lambda}{2}$ or half a wavelength separates two consecutive nodes.

Antinodes :

The amplitude has a maximum value of $2y_m$, which occurs for the values of kx that give |sin kx| = 1. Those values are

$kx = (n + 1/2)\pi$ for n = 0, 1, 2, 3,....

Substituting $k = 2\pi/\lambda$ in this equation, we get.

$$x = (n + 1/2)\frac{\lambda}{2} \text{ for } n = 0, 1, 2, 3,....$$

as the positions of maximum amplitude. These are called the **antinodes**. The antinodes are separated by $\lambda/2$ and are located half way between pairs of nodes.

For a stretched string of length L, fixed at both ends, the two ends of the ends is chosen as position x = 0, then the other end is x = L. In order that this end is a node; the length L must satisfy the condition

$$L = n\frac{\lambda}{2}, \text{ for } n = 1, 2, 3,....$$

This condition shows that standing waves on a string of length L have restricted wavelength given by

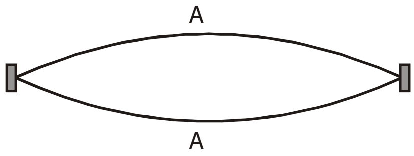
$$\lambda = \frac{2L}{n}, \text{ for } n = 1, 2, 3,.....$$

The frequencies corresponding to these wavelengths follow from Eq. as

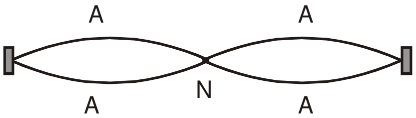
$$\nu = n\frac{v}{2L}, \text{ for } n = 1, 2, 3,.....$$

where v is the speed of travelling waves on the string. The set of frequencies given by equation are called the natural frequencies or **modes** of oscillation of the system. This equation tells us that the natural frequencies of a string are integral multiples of the lowest frequency $\nu = \frac{v}{2L}$, which corresponds to n = 1. The oscillation mode with that lowest frequency is called the fundamental mode or the first harmonic. The second harmonic or first overtone is the oscillation mode with n = 2. The third harmonic and second overtone corresponds to n = 3 and so on. The frequencies associated with these modes are often labeled as ν_1, ν_2, ν_3 and so on. The collection of all possible modes is called the harmonic series and n is called the harmonic number.

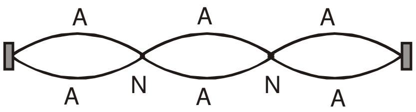
Some of the harmonic of a stretched string fixed at both the ends are shown in figure.



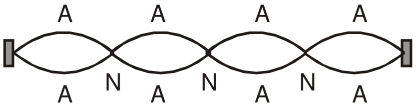
(a)
Fundamental
or first harmonic
$\nu_0 = v/2L$



(b)
second harmonic
or first overtone =
$\nu_1 = 2\nu_0 = 2v/2L$



(c)
third harmonic
or 2nd
overtone =
$\nu_2 = 3\nu_0, = 3v/2L$



(d)
fourth harmonic
or 3rd
overtone =
$\nu_3 = 3\nu_0, = 3v/2L$

Ex. 14 A middle C string on a piano has a fundamental freuquency of 262 Hz, and the A note has fndametal frquency of 440 Hz. (a) Calculate the frequencies of the next two harmonics of the C string. (b) If the strings for the A and C notes are assumed to have the same mass per unit length and the same length, determine the ratio of tensions in the two strings.

Sol. (a) Because f_1 = 262 Hz for the C string, we can use Equation to find the frequencies f_2 and f_3;

$$f_2 = 2f_1 = 524 \text{ Hz}$$
$$f_3 = 3f_1 = 786 \text{ Hz}$$

Using Equation for the two strings vibrating at their fundamental frequencies gives

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{T_A}{\mu}} \qquad f_{1C} = \frac{1}{2L}\sqrt{\frac{T_C}{\mu}}$$

$$\therefore \quad \frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}} \qquad \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440\,\text{Hz}}{262\,\text{Hz}}\right)^2$$

$$= 2.82.$$

Ex. 15 A wire having a linear mass density 5.0×10^{-3} kg/m is stretched between two rigid supports with a tension of 450 N. The wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire.

Sol. Suppose the wire vibrates at 420 Hz in its nth harmonic and at 490 Hz in its (n + 1)th harmonic.

$$420 \text{ s}^{-1} = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{....(i)}$$

and

$$490 \text{ s}^{-1} = \frac{(n+1)}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{....(ii)}$$

This gives $\dfrac{490}{420} = \dfrac{n+1}{n}$ or, $n = 6$.

Putting the value in (i),

$$420 \text{ s}^{-1} = \frac{6}{2L}\sqrt{\frac{450\,\text{N}}{5.0\times10^{-3}\,\text{kg/m}}} = \frac{900}{L} \text{ m/s}$$

or, $L = \dfrac{900}{420}$ m = 2.1 m

(b) Fixed at one end :

Standing waves can be produced on a string which is fixed at one end and whose other end is free to move in a transverse direction. Such a free end can be nearly achieved by connecting the string to a very light thread.

If the vibrations are produced by a source of "correct" frequency, standing waves are produced. If the end $x = 0$ is fixed and $x = L$ is free, the equation is again given by

$$y = 2A \sin kx \cos \omega t$$

with the boundary condition that $x = L$ is an antinode. The boundary condition that $x = 0$ is a node is automatically satisfied by the above equation. For $x = L$ to be an antinode,

$$\sin kL = \pm 1$$

or, $kL = \left(n + \frac{1}{2}\right)\pi$ or, $\frac{2\pi L}{\lambda} = \left(n + \frac{1}{2}\right)\pi$

or, $\frac{2L\nu}{v} = n + \frac{1}{2}$ or, $\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L} = \frac{n + \frac{1}{2}}{2L}\sqrt{F/\mu}$ .....

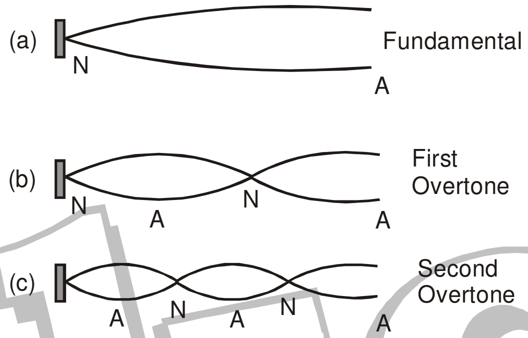
These are the normal frequencies of vibration. The fundamental frequency is obtained when $n = 0$, i.e.,

$$\nu_0 = v/4L$$

The overtone frequencies are

$$\nu_1 = \frac{3v}{4L} = 3\nu_0$$

$$\nu_2 = \frac{5v}{4L} = 5\nu_0$$



We see that all the harmonic of the fundamental are not the allowed frequencies for the standing waves. Only the odd harmonics are the overtones. Figure shows shapes of the string for some of the normal modes.

LAWS OF TRANSVERSE VIBRATIONS OF A STRING - SONOMETER WIRE

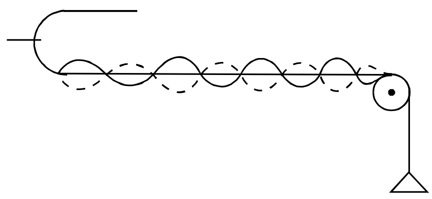
(a) Law of length $f \propto \frac{1}{L}$ so $\frac{f_1}{f_2} = \frac{L_2}{L_1}$; if T & μ are constant

(b) Law of tension $f \propto \sqrt{T}$ so $\frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$; L & μ are constant

(c) Law of mass $f \propto \frac{1}{\sqrt{\mu}}$ so $\frac{f_1}{f_2} = \sqrt{\frac{\mu_2}{\mu_1}}$; T & L are constant

MELDE'S EXPERIMENT

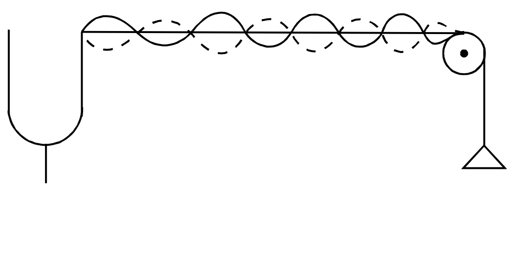
Experiment can be used to calculate unknownfrequency of tuning fork. String can be set into vibrations in two different arrangements.



(1) Transverse arrangement : Oscillations of tuning fork are set perpendicular to length of string frequency of tuning fork and of waves on string are same so if n is number of loops in string then

$$L = n\left(\frac{\lambda}{2}\right) \text{ so } f = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

(2) Oscillations of tuning fork are parallel to length of string, frequency of wave is half the frequency of tuning fork.



If n is number of loops then $f = \frac{2n}{2L}\sqrt{\frac{T}{\mu}}$.

