ANSWERSHEET (TOPIC = INTEGRAL CALCULUS) COLLECTION #2

Question Type = A.Single Correct Type

Q. 1 (B) Sol $\int_0^{16} f(t)\,dt$

consider $g(x)=\int_0^{x^4} f(t)\,dt \quad \Rightarrow \quad g(0)=0$

LMVT for g in $[0,1]$ gives some $\alpha\in(0,1)$ such that $\frac{g(1)-g(0)}{1-0}=g'(\alpha)$ … (1)

|||ly LMVT for g in $[1,2]$ gives some $\beta\in(1,2)$ such that $\frac{g(2)-g(1)}{2-1}=g'(\beta)$ …(2)

$(1)+(2) \qquad g'(\alpha)+g'(\beta)=g(2)-\underbrace{g(0)}_{\text{zero}}; \qquad$ but $g'(x)=f\left(x^4\right).4x^3$

$4\left(\alpha^3 f\left(\alpha^4\right)+\beta^3 f\left(\beta^4\right)\right)=\int_0^{16} f(t)\,dt \quad \Rightarrow \quad$ (B)]

Q. 2 (A) Sol $I=\int_{-\alpha}^{(x-\alpha)} \sin|t|\,dt$ where $2x-\alpha=t \quad \Rightarrow \quad dx=\frac{dt}{2}$

$=\frac{1}{2}\int_{-\alpha}^{0} -\sin t\,dt+\frac{1}{2}\int_{0}^{\pi-\alpha} \sin t\,dt$

$=\frac{1}{2}\cos t\Big]_{-\alpha}^{0}-\frac{1}{2}\cos t\Big]_{0}^{\pi-\alpha}=\frac{1}{2}[1-\cos\alpha]-\frac{1}{2}[-\cos\alpha-1]$

$=\frac{1}{2}(1-\cos\alpha)+\frac{1}{2}(1+\cos\alpha)=1$ Ans.]

Q. 3 (A) Sol $a_n=\int_0^{\pi/2}(1-\sin t)^n \sin 2t\,dt$

Let $1-\sin t=u \quad \Rightarrow \quad -\cos t\,dt=du$

$=2\int_0^1 u^n(1-u)\,du=2\left(\int_0^1 u^u\,du-\int_0^1 u^{n+1}\,du\right)=2\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

hence $\frac{a_n}{n}=2\left(\frac{1}{n(n+1)}-\frac{1}{n(n+2)}\right)$

$\lim_{n\to\infty}\sum_{n=1}^{n}\frac{a_n}{n}=2\left(\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{2}\sum\left(\frac{1}{n}-\frac{1}{n+2}\right)\right)=2\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

$=2(1)-\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+......\right]=2-\frac{3}{2}=\frac{1}{2}$ Ans.]

Q. 4 (C) Sol $\left[x^3 - 4014x^2 + (2007)^2 x + \frac{x}{2008}\right]_0^{2008}$

$= (2008)^3 - 4014(2008)^2 + \left((2007)^2 + \frac{1}{2008}\right).2008$

$= (2008)\left[(2008)^2 - (4014)(2008) + (2007)^2\right] + 1$

$= (2008)\left[(2008)^2 - 2(2007)(2008) + (2007)^2\right] + 1 = 2008\left[(2008 - 2007)^2\right] + 1$

$= 2009$ Ans.]

Q. 5 (B) Sol $y = \ln^2 x - 1$

$y' = \frac{2\ln x}{x} = 0 \qquad \Rightarrow \qquad x = 1$

$x > 1,\ y\uparrow$ and $0 < x < 1,\ y$ is $\downarrow$

$A = \left|\int_{1/e}^{e} (\ln^2 - 1)\,dx\right|$

$= \left| x\ln^2 x\Big]_{1/e}^{e} - 2\int_{1/e}^{e}\left(\frac{\ln x}{x}\right).xdx - \left(e - \frac{1}{e}\right)\right|$

$= \left|\left(e - \frac{1}{e}\right) - 2\int_{1/e}^{e}\left(\frac{\ln x}{x}\right).xdx - \left(e - \frac{1}{e}\right)\right|$

$= \left|-2\left[x\ln x\Big]_{1/e}^{e} - \int_{1/e}^{e} dx\right]\right| = \left|-2\left[\left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right)\right]\right| = \left|\frac{4}{e}\right| = \frac{4}{e}$ **Ans.]**

Q. 6 (A) Sol Consider $I = \int_0^1 (by + a(1-y))^x\,dy$

$= \int_0^1 (a + (b-a)y)^x\,dy = \left[\frac{(a + (b-a)y)^{x+1}}{(x+1)} \cdot \frac{1}{b-a}\right]_0^1$

$I = \frac{1}{(x+1)(b-a)}\left(b^{x+1} - a^{x+1}\right) = \frac{1}{(x+1)}\left(\frac{b^{x+1} - a^{x+1}}{b-a}\right)$

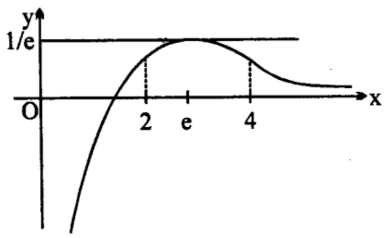
now $L = \lim_{x\to 0}\left(\frac{b^{x+1} - a^{x+1}}{b-a}\right)^{1/x} \cdot \left(\frac{1}{(x+1)}\right)^{1/x}$

$= \underbrace{\lim_{x\to 0}\left(\frac{1}{(x+1)}\right)^{1/x}}_{1^\infty} \cdot \underbrace{\lim_{x\to 0}\left(\frac{b^{x+1} - a^{x+1}}{b-a}\right)^{1/x}}_{1^\infty} \quad \left(\begin{array}{c} \lim_{x\to 0}(x+1)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}(x+1-1)} = e \\ \Rightarrow \quad \frac{1}{(x+1)^{1/x}} = \frac{1}{e} \end{array}\right)$

$$\therefore \qquad L = \frac{1}{e}.\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{l}$$

$$\text{now,} \quad l = e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{b^{x+1}-a^{x+1}-b+a}{b-a}\right)} = e^{\frac{1}{b-a}\lim\limits_{x\to 0}\frac{1}{x}\frac{b(b^x-1)-a(a^x-1)}{x}} = e^{\frac{1}{b-a}(b/n\ b-a/n\ a)}$$

$$= e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}} = \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$



$$\therefore \qquad L = \frac{1}{e}.\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \qquad \text{Ans.]}$$

Q. 7 (C) Sol $f'(x) = \frac{1}{\sqrt{1+g^2(x)}}.g'(x);$

$$f\left(\frac{\pi}{2}\right) = \frac{g'(\pi/2)}{\sqrt{1+g^2(\pi/2)}}; \qquad g\left(\frac{\pi}{2}\right) = 0$$

$$= g'\left(\frac{\pi}{2}\right)$$

Now $g(x) = \left[1+\sin\left(\cos^2 x\right)\right](-\sin x)$

$$g'\left(\frac{\pi}{2}\right) = 1(-1) = -1$$

hence $f'\left(\frac{\pi}{2}\right) = -1$ as $h'(0^+) = -1 \quad \Rightarrow \quad$ (C)]

Q. 8 (A) Sol $I = \int\limits_{\pi/2}^{3\pi/2} \underset{II}{1}.\underbrace{f(x)}_{I}dx = xf(x)\Big]_{\pi/2}^{3\pi/2} - \int\limits_{\pi/2}^{3\pi/2} f'(x).xdx$

$$= \frac{3\pi}{2}f'\left(\frac{3\pi}{2}\right) - \frac{\pi}{2}f\left(\frac{\pi}{2}\right) - \underbrace{\int\limits_{\pi/2}^{3\pi/2}\frac{\cos x}{x}.xdx}_{-2} = b.\frac{3\pi}{2} - a.\frac{\pi}{2} + 2 = 2 - \frac{\pi}{2}(a-3b) \qquad \text{Ans.]}$$

Q. 9 (A) Sol $I = \underbrace{\int_1^e \left[\underbrace{\left(e^x + e^{-x}\right)}_{f(x)} + \underbrace{\left(e^x - e^{-x}\right)}_{xf(x)}\right]}_{II} . \underbrace{\ln x dx}_{I}$

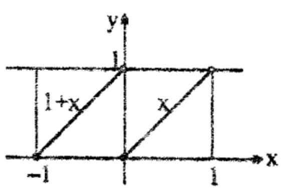
$$\left(\int \left(f(x) + xf'(x)\right)dx = xf(x)\right)$$

$$I = \ln x . xf(x)\Big]_1^e - \int_1^e \frac{xf(x)}{x}dx = ef(e) - \int_1^e \left(e^x + e^{-x}\right)dx$$

$$= e\left(e^e + e^{-e}\right) - \left[\left(e^e - e^{-e}\right) - \left(e - e^{-1}\right)\right]$$

$$= e^{e+1} + e^{1-e} - e^e + e^{-e} + e - e^{-1} \qquad \text{Ans.]}$$

Q. 10 (B) Sol $I_1 = \int_{-1}^{1} \left(\{x\}.\{x^2\} + \{x^2\}\{x^3\}\right)dx$



> Note that $\int_{-1}^{0}\left(x^3 + x^5\right)dx + \int_0^1 \left(x^3 + x^5\right)dx$
>
> $= \int_{-1}^{1}\left(x^3 + x^5\right)dx = 0$

$$I_1 = \int_{-1}^{1}\left(\{x^2\}\{x\} + \{x^3\}\right)dx$$

now, $\{x\} = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1+x & \text{if } -1 \le x < 0 \end{cases}$

Also, for $-1 < x < 1$, $\{x^2\} = x^2$

and $\{x^3\} = \begin{cases} x^3 & \text{if } 0 \le x < 1 \\ 1+x & \text{if } -1 \le x < 0 \end{cases}$

hence $I_1 = \int_{-1}^{0} x^2(1+x) + \left(1 + x^3\right)dx + \int_0^1 x^2\left(x + x^3\right)dx$

$$I_1 = \int_{-1}^{0}\left(2x^2 + x^3 + x^5\right)dx + \int_0^1 \left(x^5 + x^3\right)dx$$

$$= 2\int_{-1}^{0} x^2 dx = 2\frac{x^3}{3}\Big]_{-1}^{0} = \frac{2}{3} \qquad \textbf{Ans.}$$

Q. 11 (A) Sol $\frac{dy}{dx} - y = 1 - e^{-x}$, I.F. $= e^{-x}$

$\therefore \quad y.e^{-x} = \int \left(e^{-x} - e^{-2x}\right)dx$

$y.e^{-x} = e^{-x} + \frac{1}{2}e^{-2x} + C$

if $\quad x = 0,\ y = y_0$

$y_0 = -1 + \frac{1}{2} + C \quad \Rightarrow \quad C = y_0 + \frac{1}{2}$

$\therefore \quad y.e^{-x} = -e^{-x} + \frac{1}{2}e^{-2x} + y_0 + \frac{1}{2}$

If $\quad x \to \infty,$ then $y_0 = -\frac{1}{2}$ Ans.]

Q. 12 (A) Sol $\quad y.e^{-2x} = Axe^{-2x} + B$

$e^{-2x}.y_1 - 2ye^{-2x} = A\left(e^{-2x} - 2xe^{-2x}\right)$

Canceling e^{-2x} throughout

$y_1 - 2y = A(1-2x) \quad \ldots.(1)$

differentiating again

$y_2 - 2y_1 = -2A \quad \Rightarrow \quad A = \frac{2y_1 - y_2}{2}$

Hence substituting A in (1)

$2(y_1 - 2y) = (2y_1 - y_2)(1-2x)$

$2y_1 - 4y = 2y_1(1-2x) - (1-2x)y_2$

$(1-2x)\frac{d}{dx}\left(\frac{dy}{dx} - 2y\right) + 2\left(\frac{dy}{dx} - 2y\right) = 0$

hence $k = 2$ and $l = -2 \quad \Rightarrow \quad$ ordered pair $(k, l) \equiv (2, -2)$ Ans.]

Question Type = B.Comprehension or Paragraph

Q. 13 () Sol **Q. 1** [A]

Q. 2 [C]

Q. 3 [C]

[Sol. $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$

I.F. $= e^{\int \frac{2x}{1+x^2}dx = e^{\ln(1+x^2)}} = (1+x^2)$

$\therefore \quad y(1+x^2) = \int 4x^2 dx = \frac{4x^3}{3} + C$

Passing through $(0, 0) \Rightarrow C = 0$

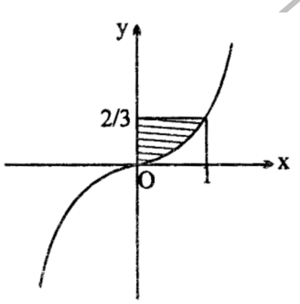
$\therefore \qquad y = \dfrac{4x^2}{3\left(1+x^2\right)}$

$$\frac{dy}{dx} = \frac{4}{3}\left[\frac{\left(1+x^2\right)3x^2 - x^3.2x}{\left(1+x^2\right)^2}\right] = \frac{4}{3}\left[\frac{3x^2+x^4}{\left(1+x^2\right)^2}\right] = \frac{4x^2\left(3+x^2\right)}{3\left(1+x^2\right)^2}$$

Hence $\dfrac{dy}{dx} > 0 \quad \forall \qquad x \neq 0;$

$\dfrac{dy}{dx} = 0$ at $x = 0$ and it does not change sign p $x = 0$ is the point of inflection Ans.

$y = f\left(x\right)$ is increasing for all $x \in R$



$x \to \infty; \;\; y \to \infty \qquad ; \qquad x \to -\infty; \;\; y \to -\infty$

Area enclosed by $y = f^{-1}\left(x\right)$, x-axis and ordinate at $x = \dfrac{2}{3}$

$$A = \frac{2}{3} - \frac{4}{3}\int_0^1 \frac{x^3}{1+x^2}dx$$

put $1+x^2 = t \qquad \Rightarrow \qquad 2xdx = dt$

$$A = \frac{2}{3} - \frac{2}{3}\int_1^2 \frac{\left(t-1\right)}{t}dt = \frac{2}{3} - \frac{2}{3}\int_1^2\left(1-\frac{1}{t}\right)dt$$

$$= \frac{2}{3} - \frac{2}{3}\left[t - \ln t\right]_1^2 = \frac{2}{3} - \frac{2}{3}\left[\left(2-\ln 2\right)-1\right]$$

$$= \frac{2}{3} - \frac{2}{3}\left[t - \ln 2\right] = \frac{2}{3}\ln 2$$ **Ans.]**

Question Type = C.Assertion Reason Type

Q. 14 (C) Sol Let $\int_0^1 f\left(t\right)dt = k$, so

$f\left(x\right) = xk + 1$, now

$\int_0^1 \left(kt+1\right)dt = k$

$\Rightarrow \qquad \dfrac{k}{2} + 1 = k$, so $k = 2$

$\therefore \quad f(x) = 2x + 1,$

Also $\int_0^3 f(x)dx = 12$

$\Rightarrow$ option (C) is correct.]

Q. 15 (D) Sol $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\sin x}$ (using King)

$2I = \int_{-\pi/4}^{\pi/4} \frac{2dx}{1-\sin^2 x} \quad \Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \frac{dx}{\cos^2 x}$

$I = 2\int_0^{\pi/4} \sec^2 x dx \neq 0 \quad \Rightarrow$ Statement-1 is false]

Q. 16 (A) Sol $\frac{dy}{y^2+1} = dx$

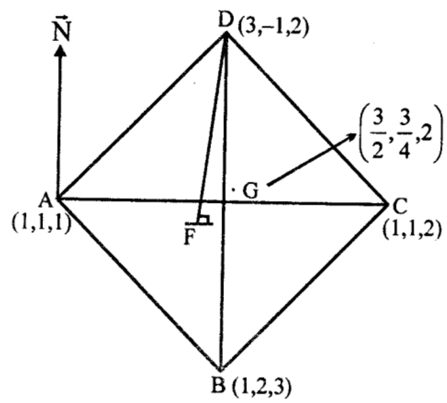
$\tan^{-1}(y) = x + C$

$\therefore \quad y = 0$ when $x = \pi$

$\Rightarrow \quad C = -\pi$

$\tan^{-1}(y) = x - \pi$

$\therefore \quad \tan^{-1}(y) \in \left(-\frac{\pi}{2}.\frac{\pi}{2}\right) \quad \Rightarrow \quad -\frac{\pi}{2} < x - \pi < \frac{\pi}{2}$



$\frac{\pi}{2} < x < \frac{3\pi}{2}$]

Question Type = D.More than one may corect type

Q. 17 () Sol B, D

[Sol. We have $f(x) = x^2 + ax^2 + bx^3$

Where $a = \int_{-1}^{1} t.f(t)\,dt$ and $b = \int_{-1}^{1} f(t)\,dt$

How $a = \int_{-1}^{1} t[(a+1)t^2 + bt^3)dt$

$a = 2b\int_0^1 t^4 dt = \frac{2b}{5}$ ....(1)

Again $b = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{1}\left((a+1)t^2 + bt^3\right)dt = 2\int_0^1 (a+1)t^2 dt$

$b = \frac{2(a+1)}{3}$ ....(2)

From (1) and (2)

$\frac{5a}{2} = \frac{2(a+1)}{3}$

$\left(\frac{5}{2} - \frac{2}{3}\right)a = \frac{2}{3} \Rightarrow \frac{11}{6}a = \frac{2}{3}$

$a = \frac{4}{11}$ and $b = \frac{10}{11}$

Hence $\int_{-1}^{1} t.f(t)\,dt = \frac{4}{11}$ and $\int_{-1}^{1} f(t)\,dt = \frac{10}{11}$

$\therefore \quad f(x) = (a+1)x^2 + bx^3$

$\left.\begin{array}{l} f(1) = (a+1) + b \\ f(-1) = (a+1) - b \end{array}\right] \Rightarrow f(1) + f(-1) = 2(a+1) = \frac{30}{11}$

and $f(1) - f(-1) = 2b = \frac{20}{11}$ $\Rightarrow$ B, D correct.

Q. 18 () Sol A, D

[Sol. Consider $f(x) = \int_{-x}^{x}\left(\underbrace{t\sin at}_{\text{even}} + \underbrace{bt}_{\text{odd}} + \underbrace{c}_{\text{even}}\right)dt = 2\int_0^x (t\sin at + c)\,dt$

$= 2\left[-t\frac{\cos at}{a}\Big|_0^x + \int_0^x \frac{\cos at}{a}dt + ct\,\Big|_0^x\right]$ (using I.B.P.)

$= 2\left[\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax + cx\right]$

$\therefore \quad \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} 2\left[-\frac{\cos ax}{a} + \frac{\sin ax}{a.ax} + c\right]$

$= 2\left[-\frac{1}{a} + \frac{1}{a} + c\right] = 2c]$

Question Type = E.Match the Columns

Q. 19 () Sol (A) R; (B) S; (C) P

[Sol. (A) $\theta_1+\theta_2=\frac{\pi}{2}$

$\therefore$ $I=\int_{\theta_1}^{\theta_2}\frac{d\theta}{1+\tan\left(\frac{\pi}{2}-\theta\right)}=\int_{\theta_1}^{\theta_2}\frac{\tan\theta d\theta}{1+\tan\theta}$ (using King)

$2I=\int_{\theta_1}^{\theta_2}d\theta=\theta_2-\theta_1=\frac{1002\pi}{2008}\Rightarrow$ $I=\frac{501\pi}{2008}$ Ans. $\Rightarrow$ (R)

(B) $I=\int_0^1\left[g^2(x).\frac{\{f(x).g'(x)+f'(x).g(x)\}}{g^2(x)}+\frac{\{g(x).f'(x)+f(x).g'(x)\}}{g^2(x)}\right]dx$

$=\int_0^1\left[\left\{\frac{d}{dx}f(x).g(x)\right\}+\left\{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\right\}\right]dx=f(x).g(x)+\left.\frac{f(x)}{g(x)}\right]_0^1$

$=\left[f(1).g(1)+\frac{f(1)}{g(1)}\right]-\left[f(0).g(0)+\frac{f(0)}{g(0)}\right]$

$=\left[\frac{2009}{2}+\frac{2009}{2}\right]-\left[(0)-(0)\right]=2009$ Ans. $\Rightarrow$ (S)

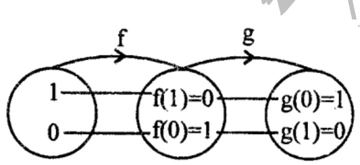
(C) Consider $y=f(x)=\left(1-x^{n+1}\right)^{1/n}$ where $=2007$

$\Rightarrow$ $x=f^{-1}(y)=g(y)$ (say) (note that $f(1)=0$ and $f(0)=1$ and f is monotonic decreasing)

$\therefore$ $y^n=1-x^n+1;$ $x^{n+1}=1-y^n;$ $x=\left(1-y^n\right)\frac{1}{n+1};$

$f^{-1}(y)=\left(1-y^n\right)^{\frac{1}{n+1}};$ $g(y)=\left(1-y^n\right)^{\frac{1}{n+1}};$ $g(x)=\left(1-x^n\right)^{\frac{1}{n+1}}=\left(1-x^{2007}\right)\frac{1}{2008}$

Hence the two function appearing as integrand are inverse of each other.



$\therefore$ $I=\int_0^1 f(x)dx-\int_0^1 g(y)dy$

but $y=f(x)$ $\Rightarrow$ $dy=f'(x)dx$

and $x=g(y)$

$$I = \int_0^1 f(x)\,dx - \int_1^0 xf'(x)\,dx = \int_0^1 \left(f(x) + x\left(f'(x)\right)dx\right) = xf(x)\Big]_0^1$$

$= f(1) - 0 = 0$ Ans. $\Rightarrow$ (P)]]