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STUDY PACKAGE

Subject : Mathematics

Topic : DIFFRENTIATION

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Differentiation

A. First Principle Of Differentiation

1. The derivative of a given function f at a point x = a on its domain is defined as:

$\underset{h \to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

i.e. $f'(a) = \underset{x \to a}{\text{Limit}} \dfrac{f(x)-f(a)}{x-a}$, provided the limit exists.

2 If x and x + h belong to the domain of a function f defined by y = f(x), then

$\underset{h \to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ if it exists, is called the Derivative of f at x & is denoted by $f'(x)$ or $\dfrac{dy}{dx}$. i.e., $f'(x) = \underset{h \to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ This method of differentiation is also called ab-initio method or first principle.

Solved Example # 1 Find derivative of following functions by first principle

(i) $f(x) = x^2$ (ii) $f(x) = \tan x$ (iii) $f(x) = e^{\sin x}$

Solution (i) $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h \to 0} \dfrac{2xh + h^2}{h} = 2x.$

(ii) $f'(x) = \lim\limits_{h \to 0} \dfrac{\tan(x+h) - \tan x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\tan(x+h-x)[1+\tan x \tan(x+h)]}{h} = \lim\limits_{h \to 0} \dfrac{\tan h}{h} . (1 + \tan^2 x) = \sec^2 x.$

(iii) $f'(x) = \lim\limits_{h \to 0} \dfrac{e^{\sin(x+h)} - e^{\sin x}}{h}$

$= \lim\limits_{h \to 0} e^{\sin x} \dfrac{\left[e^{\sin(x+h)-\sin x} - 1\right]}{\sin(x+h) - \sin x} \left(\dfrac{\sin(x+h) - \sin x}{h}\right)$

$= e^{\sin x} \lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h} = e^{\sin x} \cos x$

3. Differentiation of some elementary functions

	f(x)	f′(x)	
1.	x^n	nx^{n-1}	$(x \in R, n \in R)$
2.	a^x	$a^x \ell n\, a$	
3.	$\ell n \lvert x \rvert$	$\dfrac{1}{x}$	
4.	$\log_a x$	$\dfrac{1}{x\, \ell n\, a}$	
5.	sin x	cos x	
6.	cos x	– sin x	
7.	sec x	sec x tan x	
8.	cosec x	– cosec x cot x	
9.	tan x	$\sec^2 x$	
10.	cot x	– cosec x	

4. Basic Theorems

1. $\dfrac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$

2. $\dfrac{d}{dx}(k\, f(x)) = k \dfrac{d}{dx} f(x)$

3. $\dfrac{d}{dx}(f(x) . g(x)) = f(x)\, g'(x) + g(x)\, f'(x)$

4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\, f'(x) - f(x) g'(x)}{g^2(x)}$

5. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$

This rule is also called the chain rule of differentiation and can be written as $\dfrac{dy}{dx} = \dfrac{dy}{dz} . \dfrac{dz}{dx}$

Note that an important inference obtained from the chain rule is that $\dfrac{dy}{dy} = 1 = \dfrac{dy}{dx} . \dfrac{dx}{dy}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{dx/dy}$

another way of expressing the same concept is by considering y = f(x) and x = g(y) as inverse functions of each other.

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{dx}{dy} = g'(y) \quad \Rightarrow \quad g'(y) = \frac{1}{f'(x)}$$

Solved Example # 2

Find the differential of the following functions with respect to x.

(i) $f(x) = e^{\sin x}$ (ii) $f(x) = \sqrt{\sin(2x+3)}$ (iii) $f(x) = \frac{x}{1+x^2}$ (iv) $f(x) = x \,.\, \sin x$

Solution. (i) $f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$

(ii) $f(x) = \sqrt{\sin(2x+3)}$

$= \frac{1}{2\sqrt{\sin(2x+3)}} \cdot \frac{d}{dx}(\sin(2x+3)) = \frac{\cos(2x+3)}{\sqrt{\sin(2x+3)}}$

(iii) $f(x) = \frac{x}{1+x^2}$

$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

(iv) $f(x) = x \sin x$

$f'(x) = x.\cos x + \sin x$

Solved Example # 3 If f(x) = sin (x + tanx) then find value of f′(0).

Solution. $f'(x) = \cos(x + \tan x)(1 + \sec^2 x)$ $f'(0) = 2$

Self Practice Problems :

1. Find the derviative of following functions using first principle.

(i) $f(x) = x \sin x$ (ii) $f(x) = \sin^2 x$

Ans. (i) $x \cos x + \sin x$ (ii) $2\sin x \cos x$

2. Evaluate if f′(5) = 7, then $\lim_{t \to 0} \frac{f(5+t) - f(5-t)}{2t}$ **Ans.** 7.

3. Differentiate the following functions

(i) $(1 + 3x^2)(2x^3 - 1)$ (ii) $\frac{(x-1)}{(x-2)(x-3)}$ (iii) $\sqrt{1+x^2}$

(iv) $\sqrt{\frac{1+x}{1-x}}$ (v) $\cos^3 x \sin x$ (vi) $x e^x \sin x$

(vii) $\frac{\sin x}{1+\cos x}$ (viii) $\ell n(\sin x - \cos x)$

Ans. (i) $6x(5x^3 + x - 1)$ (ii) $\frac{-x^2+2x+1}{(x-2)^2(x-3)^2}$ (iii) $\frac{x}{\sqrt{1+x^2}}$ (iv) $\frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$

(v) $\cos^4 x - 3\cos^2 x \sin^2 x$ (vi) $e^x((\sin x + \cos x)x + \sin x)$ (vii) $\frac{1}{2}\sec^2\frac{x}{2}$ (viii) $\frac{\cos x + \sin x}{\sin x - \cos x}$

B. Derivative Of Inverse Trigonometric Functions.

$y = \sin^{-1} x \Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow x = \sin y$

$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$ $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ $-1 < x < 1.$

Note here that $\cos y \ne \sqrt{1-\sin^2 y}$, rather $\cos y = \pm\sqrt{1-\sin^2 y}$ but for values of $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, cos y is always positive and hence the result. similarly let us find derivative of other inverse trigonometric functions.

Let $y = \tan^{-1}x$

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y \Rightarrow \frac{dx}{dy} = 1 + x^2$

$\frac{dy}{dx} = \frac{1}{1+x^2}$ $(x \in R)$

Also if $y = \sec^{-1}x$ $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\} \Rightarrow x = \sec y$

$\frac{dx}{dy} = \sec y \tan y \Rightarrow \frac{dy}{dx} = \frac{1}{x.\tan y} \Rightarrow \frac{dy}{dx} = \frac{1}{\pm x\sqrt{\sec^2 y - 1}}$

$$\frac{dy}{dx} = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \sec y > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \sec y < -1 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}} \qquad x \in (-\infty, -1) \cup (1, \infty)$$

results for the derivative of inverse trigonometric functions can be summarized as :

f(x)	f′(x)	
$\sin^{-1}x$	$\dfrac{1}{\sqrt{1-x^2}}$	; $\lvert x\rvert < 1$
$\cos^{-1}x$	$\dfrac{-1}{\sqrt{1-x^2}}$	; $\lvert x\rvert < 1$
$\tan^{-1}x$	$\dfrac{1}{1+x^2}$	; $x \in R$
$\cot^{-1}x$	$\dfrac{-1}{1+x^2}$	; $x \in R$
$\sec^{-1}x$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	; $\lvert x\rvert > 1$
$\text{cosec}^{-1}x$	$\dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	; $\lvert x\rvert > 1$

Solved Example # 4 If $f(x) = \ell n(\sin^{-1} x^2)$ find $f'(x)$

Solution. $f'(x) = \dfrac{1}{(\sin^{-1}x^2)} \cdot \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \dfrac{2x}{(\sin^{-1}x^2)\sqrt{1-x^4}}$

Solved Example # 5 If $f(x) = 2x \sec^{-1}x - \text{cosec}^{-1}(x)$ then find $f'(-2)$

Solution. $f'(x) = 2\sec^{-1}(x) - \dfrac{2x}{|x|\sqrt{x^2-1}} + \dfrac{1}{|x|\sqrt{x^2-1}}$

$f'(-2) = 2.\sec^{-1}(-2) + \dfrac{2}{\sqrt{3}} + \dfrac{1}{2\sqrt{3}}$ $\qquad f'(-2) = \dfrac{4\pi}{3} + \dfrac{5}{2\sqrt{3}}$.

C. Methods Of Differentiation

1. Logrithmic Differentiation

The process of taking logarithm of the function first and then differentiate is called **Logarithmic Differentiation**. It is useful if

(i) a function is the product or quotient of a number of functions OR

(ii) a function is of the form $[f(x)]^{g(x)}$ where f & g are both derivable,

Solved Example # 6 If $y = x^x$ find $\dfrac{dy}{dx}$

Solution. $\ell n\, y = x\, \ell n\, x \Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = x \cdot \dfrac{1}{x} + \ell n\, x \Rightarrow \dfrac{dy}{dx} = x^x(1 + \ell n\, x)$

Solved Example # 7 If $y = (\sin x)^{\ell n\, x}$, find $\dfrac{dy}{dx}$

Solution. $\ell n\, y = \ell n\, x \cdot \ell n(\sin x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}\ell n(\sin x) + \ell n\, x. \dfrac{\cos x}{\sin x} \Rightarrow \dfrac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\dfrac{\ell n(\sin x)}{x} + \cot x\ \ell n\, x\right]$

Solved Example # 8 If $y = \dfrac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\dfrac{dy}{dx}$

Solution. $\ell n\, y = \dfrac{1}{2}\ell n\, x + \dfrac{2}{3}\ell n(1-2x) - \dfrac{3}{4}\ell n(2-3x) - \dfrac{4}{5}\ell n(3-4x)$

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} - \dfrac{4}{3(1-2x)} + \dfrac{9}{4(2-3x)} + \dfrac{16}{5(3-4x)}$

$\dfrac{dy}{dx} = y\left(\dfrac{1}{2x} - \dfrac{4}{3(1-2x)} + \dfrac{9}{4(2-3x)} + \dfrac{16}{5(3-4x)}\right)$

2. Implicit differentiation

If $f(x, y) = 0$, is an implicit function then in order to find dy/dx, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx.

Solved Example # 9 If $x^3 + y^3 = 3xy$ find $\dfrac{dy}{dx}$

Solution. Differentiation both sides w.r.t.x, we get

$3x^2 + 3y^2\dfrac{dy}{dx} = 3x\dfrac{dy}{dx} + 3y \qquad \dfrac{dy}{dx} = \dfrac{y-x^2}{y^2-x}$

Note that above result holds only for points where $y^2 - x \neq 0$

Solved Example # 10 If $x^y = e^{x-y}$, then find $\frac{dy}{dx}$

Solution.
Taking log on both sides
$y\ \ell n\ x = (x - y)$ ....(i)
differentiating w.r.t x, we get

$$\frac{y}{x} + \ln x\frac{dy}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\ell n\ x} \Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\ell n\ x)}$$

Solved Example # 11 If $x^y + y^x = 2$ then find $\frac{dy}{dx}$

Solution. $u + v = 2 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$

where $u = x^y$ & $v = y^x$

$\Rightarrow \ell n\ u = y\ \ell n\ x$ & $\ell n\ v = x\ \ell n\ y$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell n\ x\frac{dy}{dx}$ & $\frac{1}{v}\frac{dv}{dx} = \ell n\ y + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell n\ x\frac{dy}{dx}\right)$ & $\frac{dv}{dx} = y^x\left(\ell n\ y + \frac{x}{y}\frac{dy}{dx}\right)$

$$\Rightarrow x^y\left(\frac{y}{x} + \ell n\ x\frac{dy}{dx}\right) + y^x\left(\ell n\ y + \frac{x}{y}\frac{dy}{dx}\right) = 0. \Rightarrow \frac{dy}{dx} = -\frac{\left(y^x\ell n\ y + x^y.\frac{y}{x}\right)}{\left(x^y\ell n\ x + y^x.\frac{x}{y}\right)}$$

Self Practice Problems

1. Differentiate the following functions : (i) $y = \sec^{-1}(x^2)$ (ii) $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

(iii) $y = \left(1+\frac{1}{x}\right)^x$ (iv) $y = e^{x^x}$ (v) $y = (\ln x)^x + (x)^{\sin x}$

2. Find $\frac{dy}{dx}$ if
(i) $y = \cos(x + y)$ (ii) $x^{2/3} + y^{2/3} = a^{2/3}$ (iii) $x = y\ \ell n\ (x - y)$

3. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\ell n\ x}{(1+\ell n\ x)^2}$.

4. If $\frac{x}{x-y} = \log\frac{a}{x-y}$, prove that $\frac{dy}{dx} = 2 - \frac{x}{y}$.

Ans. **1.** (i) $\frac{2}{x\sqrt{x^4-1}}$ (ii) $\frac{1}{1+x^2}$ (iii) $\left(1+\frac{1}{x}\right)^x\left[\ell n\left(1+\frac{1}{x}\right)+\frac{1}{1+x}\right]$

(iv) $x^x.\ e^{x^x}(\ell nx + 1)$ (v) $\left(\ell n\ (\ell nx) + \left(\frac{1}{\ell nx}\right)\right)(\ell n\ x)^x + x^{\sin x}\left(\frac{\sin x}{x} + \cos x\ \ell nx\right)$

2. (i) $\frac{-\sin(x+y)}{1+\sin(x+y)}$ (ii) $-\left(\frac{y}{x}\right)^{1/3}$ (iii) $\frac{y(x-y)}{x(x+y)}$

3. Differentiation using substitution
Following substitutions are normally used to sumplify these expression.

(i) $\sqrt{x^2+a^2} \Rightarrow x = a\tan\theta$ or $a\cot\theta$

(ii) $\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$ or $a\cos\theta$

(iii) $\sqrt{x^2-a^2} \Rightarrow x = a\sec\theta$ or $a\,\text{cosec}\,\theta$

(iv) $\sqrt{\frac{x+a}{a-x}} \Rightarrow x = a\cos\theta$

Solved Example # 12 : Differentiate $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$.

Solution. Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \tan^{-1}\left(\frac{|\sec\theta|-1}{\tan\theta}\right)$ $[\ |\sec\theta| = \sec\theta\ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)]$

$\Rightarrow \quad y = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right) \qquad \Rightarrow \quad y = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$\Rightarrow \quad y = \dfrac{\theta}{2} \qquad [\tan^{-1}(\tan x) = x \text{ for } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)]$

$\Rightarrow \quad y = \dfrac{1}{2}\tan^{-1} x \qquad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$

Solved Example # 13 : Find $\dfrac{dy}{dx}$ where $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Solution. $x = \cos\theta$

$\theta = \cos^{-1}(x) \quad ; \quad \theta \in [0, \pi]$

$\Rightarrow \quad y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right) \qquad \Rightarrow \quad y = \tan^{-1}\left(\dfrac{\sqrt{2}\cos\dfrac{\theta}{2}-\sqrt{2}\sin\dfrac{\theta}{2}}{\sqrt{2}\cos\dfrac{\theta}{2}+\sqrt{2}\sin\dfrac{\theta}{2}}\right)$

$\Rightarrow \quad y = \tan^{-1}\left(\dfrac{1-\tan\dfrac{\theta}{2}}{1+\tan\dfrac{\theta}{2}}\right) \qquad \Rightarrow \quad y = \dfrac{\pi}{4} - \dfrac{\theta}{2}$

$\Rightarrow \quad y = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x \qquad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-x^2}}$

Note that $\sqrt{1+\cos\theta} = \left|\sqrt{2}\cos\dfrac{\theta}{2}\right|$ but for $\dfrac{\theta}{2} \in \left(0, \dfrac{\pi}{2}\right)$, $\left|\sqrt{2}\cos\dfrac{\theta}{2}\right| = \sqrt{2}\cos\dfrac{\theta}{2}$

Also $\tan^{-1}(\tan x) = x$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Solved Example # 14 If $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ then find

(i) $f'(2)$ (ii) $f'\left(\dfrac{1}{2}\right)$ (iii) $f'(1)$

Solution. $x = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}(x) \quad ; \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2} \quad \Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \dfrac{\pi}{2} < 2\theta < \pi \\ 2\theta & -\dfrac{\pi}{2} \le 2\theta \le \dfrac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\dfrac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1}x & x > 1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases} \Rightarrow f'(x) = \begin{cases} -\dfrac{2}{1+x^2} & x > 1 \\ \dfrac{2}{1+x^2} & -1 < x < 1 \\ \dfrac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\dfrac{2}{5}$ (ii) $f'\left(\dfrac{1}{2}\right) = \dfrac{8}{5}$ (iii) $f'(1^+) = -1$ & $f'(1^-) = +1$

$\Rightarrow$ $f'(1)$ does not exist.

Aliter Above problem can also be solved without any substution also, but in a little tedious way.

$f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$f'(x) = \dfrac{1}{\sqrt{1-\dfrac{4x^2}{(1+x^2)^2}}} \cdot \dfrac{2\{(1+x^2)-2x^2\}}{(1+x^2)^2}$

$= \dfrac{(1+x^2)}{\sqrt{(1-x^2)^2}} \cdot \dfrac{2(1-x^2)}{(1+x^2)^2}$

$f'(x) = \dfrac{2}{(1+x^2)} \cdot \dfrac{(1-x^2)}{|1-x^2|}$ thus $f'(x) = \begin{cases} \dfrac{2}{1+x^2} & |x| < 1 \\ \dfrac{-2}{1+x^2} & |x| > 1 \end{cases}$

Solved Example # 15 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

Solution. Put $x = \sin\alpha \quad \Rightarrow \quad \alpha = \sin^{-1}(x)$

$y = \sin\beta \quad \Rightarrow \quad \beta = \sin^{-1}(y)$

$\Rightarrow \quad \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$\Rightarrow \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 2a\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \quad \Rightarrow \quad \cot\left(\frac{\alpha-\beta}{2}\right) = a$

$\Rightarrow \quad \alpha - \beta = 2\cot^{-1}(a) \quad \Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}(a)$

differentiating w.r.t to x.

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Aliter Using implicit differention.

$$\frac{-x}{\sqrt{1-x^2}} - \frac{y}{\sqrt{1-y^2}}\frac{dy}{dx} = a\left(1-\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(a-\frac{y}{\sqrt{1-y^2}}\right)\frac{dy}{dx} = a + \frac{x}{\sqrt{1-x^2}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a+\frac{x}{\sqrt{1-x^2}}}{a-\frac{y}{\sqrt{1-y^2}}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}+\frac{x}{\sqrt{1-x^2}}}{\frac{\sqrt{1-x^2}+\sqrt{1-y^2}}{x-y}-\frac{y}{\sqrt{1-y^2}}}$$

$$\frac{dy}{dx} = \frac{(1-x^2)+\sqrt{(1-x^2)(1-y^2)}+x^2-xy}{\sqrt{(1-x^2)(1-y^2)}+(1-y^2)-xy+y^2}\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \frac{1+\sqrt{(1-x^2)(1-y^2)}-xy}{1+\sqrt{(1-x^2)(1-y^2)}-xy}\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ **Hence proved**

4. Parametric Differentiation If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Solved Example # 16 If $x = a\cos^3 t$ and $y = a\sin^3 t$. Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3a\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\tan t$$

Solved Example # 17 If $y = a\cos t$ and $x = a(t - \sin t)$ find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \frac{-a\sin t)}{a(1-\cos t)}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1.$$

5. Derivative of one function with respect to another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

Solved Example # 18

Find derivative of $y = \ell n\, x$ with respect to $z = e^x$.

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1}{xe^x}$$

Self Practice Problems :

1. Find $\frac{dy}{dx}$ when
   (i) $x = a(\cos t + t\sin t)$ & $y = a(\sin t - t\cos t)$
   (ii) $x = a\left(\frac{1-t^2}{1+t^2}\right)$ & $y = b\cdot\left(\frac{2t}{1+t^2}\right)$

   Ans. (i) $\tan t$ (ii) $\frac{(t^2-1)b}{2at}$

2. If $y = \sin^{-1}\left(\frac{x^2}{\sqrt{x^4+a^4}}\right)$ then prove that $\frac{dy}{dx} = \frac{2xa^2}{x^4+a^4}$.

3. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ then prove that $\frac{dy}{dx} = \frac{2}{1+x^2}$ $(|x| \neq 1)$

4. If $u = \sin(m \cos^{-1}x)$ and $v = \cos(m \sin^{-1} x)$ then prove that $\frac{du}{dv} = \sqrt{\frac{1-u^2}{1-v^2}}$.

D. Derivatives of Higher Order

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w. r. t. x.

If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ It is also denoted by $f'''(x)$ or y'''.

Solved Example # 19
If $y = x^3 \ell n\, x$ then y'' and y'''

Solution. $y' = 3x^2 \ell n\, x + x^3 \frac{1}{x}$

$y' = 3x^2 \ell n\, x + x^2$

$y'' = 6x\, \ell n\, x + 3x^2 . \frac{1}{x} + 2x$

$y'' = 6x\, \ell n\, x + 5x$

$y''' = 6\, \ell n\, x + 11$

Solved Example # 20 If $y = \left(\frac{1}{x}\right)^x$ then find $y''(1)$

Solution.

$\ell n\, y = -x\, \ell n\, x$ when $x = 1 \Rightarrow y = 1$

$\Rightarrow \frac{y'}{y} = -(1 + \ell n\, x) \Rightarrow y' = -y(1 + \ell n\, x)$ ......(i)

again diff. w.r.t. to x,

$y'' = -y'(1 + \ell n\, x) - y . \frac{1}{x} \Rightarrow y'' = y(1 + \ln x)^2 - \frac{y}{x}$ (using (i)) $\Rightarrow y''(1) = 0$

It must be carefully noted that in case of parametric functions

although $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ **but** $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{dx^2/dt^2}$ **rather** $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$

which on applying chain rule can be resolved as

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{dy/dt}{dx/dt}\right\} . \frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}.\frac{d^2y}{dt^2} - \frac{dy}{dt}.\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2} . \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\left[\frac{dx}{dt}.\frac{d^2y}{dt^2} - \frac{dy}{dt}.\frac{d^2x}{dt^2}\right]}{\left(\frac{dx}{dt}\right)^3}$$

Solved Example # 21 If $x = t + 1$ and $y = t^2 + t^3$ then find $\frac{d^2y}{dx^2}$.

Solution. $\frac{dy}{dt} = 2t + 3t^2$; $\frac{dx}{dt} = 1$

$\Rightarrow \frac{dy}{dx} = 2t + 3t^2 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}(2t + 3t^2) . \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = 2 + 6t.$

Solved Example # 22 If $x = 2 \cos t - \cos 2t$ and $y = 2 \sin t - \sin 2t$ then find value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

Solution. $\frac{dy}{dt} = 2 \cos t - 2 \cos 2t$ $\frac{dx}{dt} = 2 \sin 2t - 2 \sin t$

$$\frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = \frac{2\sin\frac{3t}{2}.\ \sin\frac{t}{2}}{2\cos\frac{3t}{2}.\ \sin\frac{t}{2}}.$$

$\Rightarrow \frac{dy}{dx} = \tan\frac{3t}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{3t}{2}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\tan\frac{3t}{2}\right). \frac{dt}{dx}$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}.\sec^2\frac{3t}{2}}{2(\sin 2t - \sin t)} \qquad \Rightarrow \qquad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -\frac{3}{2}$$

Solved Example # 23 Find second order derivative of y= sin x with respect to z = e^x.

Solution. $\Rightarrow \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{e^x}$

$$\Rightarrow \quad \frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{\cos x}{e^x}\right) \qquad \Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\cos x}{e^x}\right).\frac{dx}{dz}$$

$$= \frac{-e^x\sin x - \cos x e^x}{(e^x)^2} \cdot \frac{1}{e^x}$$

$$\frac{d^2y}{dz^2} = -\frac{(\sin x + \cos x)}{e^{2x}}$$

Solved Example # 24: y = f(x) and x =g(y) are inverse functions of each other than express g′(y) and g′′(y) in terms of derivative of f(x).

Solution. $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$\Rightarrow \quad g'(y) = \frac{1}{f'(x)}$ ....(i) again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{f'(x)}\right).\frac{dx}{dy}$$

$$= -\frac{f''(x)}{f'(x)^2}.g'(y)$$

$\Rightarrow \quad g''(y) = -\frac{f''(x)}{f'(x)^3}$ ....(ii) which can also be remembered as

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}.$$

Solved Example # 25 y = sin (sinx) then prove that $y'' + (\tan x)\, y' + y\cos^2 x = 0$

Solution. Such expression can be easily proved using implict differention.

$\Rightarrow \quad y' = \cos(\sin x)\cos x \quad \Rightarrow \quad \sec x.y' = \cos(\sin x)$

again differentiating w.r.t x, we can get

$\sec x\, y'' + y' \sec x \tan x = -\sin(\sin x)\cos x$

$\Rightarrow \quad \tan x\, y' = -y\,.\cos^2 x \quad \Rightarrow \quad y'' + (\tan x)\, y' + y\cos^2 x = 0$

Self Practice Problems :

1. If $y = \frac{\ell n\, x}{x}$ then find $\frac{d^2y}{dx^2}$ **Ans.** $\frac{2\ell n\, x - 3}{x^3}$

2. Prove that y = x + tan x satisfies the differentiation equation

$$\cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 0.$$

3. If x = a (cosθ + θ sin θ) and y = a(sin θ – θ cosθ) then find $\frac{d^2y}{dx^2}$. **Ans.** $\frac{\sec^3\theta}{a\theta}$

4. Find second derivative of ℓnx with respect to sin x. **Ans.** $\frac{x\sin x - \cos x}{x^2\cos^3 x}$

5. if $y = e^{-x}(A\cos x + B\sin x)$, prove that

$$\frac{d^2y}{dx^2} + 2\,.\,\frac{dy}{dx} + 2y = 0.$$

Solved Example # 26 If $y = (\tan^{-1}x)^2$ then prove that $(1 + x^2)^2\frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} = 2.$

Solution.

$$\frac{dy}{dx} = \frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2)\frac{dy}{dx} = 2\tan^{-1}(x) \qquad \Rightarrow \qquad (1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{(1+x^2)}$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} = 2$$

11. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then $F'(x)$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

12. L' Hospital's Rule:

If f(x) & g(x) are functions of x such that:

(i) $\underset{x\to a}{\text{Limit}}\ f(x) = 0 = \underset{x\to a}{\text{Limit}}\ g(x)$ OR $\underset{x\to a}{\text{Limit}}\ f(x) = \infty = \underset{x\to a}{\text{Limit}}\ g(x)$ &

(ii) Both f(x) & g(x) are continuous at x = a &

(iii) Both f(x) & g(x) are differentiable at x = a &

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a, Then

$\underset{x\to a}{\text{Limit}}\ \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Limit}}\ \frac{f'(x)}{g'(x)} = \underset{x\to a}{\text{Limit}}\ \frac{f''(x)}{g''(x)}$ & so on till indeterminant form vanishes

QUESTION BANK ON METHOD OF DIFFERENTIATION

***Select the correct alternative* : (Only one is correct)**

Q.1 If g is the inverse of f & $f'(x) = \frac{1}{1+x^5}$ then $g'(x) =$

(A) $1 + [g(x)]^5$ (B) $\frac{1}{1+[g(x)]^5}$ (C) $-\frac{1}{1+[g(x)]^5}$ (D) none

Q.2 If $y = \tan^{-1}\left(\frac{\ell n \frac{e}{x^2}}{\ell n e x^2}\right) + \tan^{-1}\frac{3+2\,\ell n x}{1-6\,\ell n x}$ then $\frac{d^2y}{dx^2} =$

(A) 2 (B) 1 (C) 0 (D) −1

Q.3 If $y = f\left(\frac{3x+4}{5x+6}\right)$ & $f'(x) = \tan x^2$ then $\frac{dy}{dx} =$

(A) $\tan x^3$ (B) $-2\tan\left[\frac{3x+4}{5x+6}\right]^2 \cdot \frac{1}{(5x+6)^2}$ (C) $f\left(\frac{3\tan x^2+4}{5\tan x^2+6}\right)\tan x^2$ (D) none

Q.4 If $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$ & $\frac{dy}{dx} = \frac{1}{2\sqrt{x(1-x)}} + p$, then p =

(A) 0 (B) $\sin^{-1}x$ (C) $\sin^{-1}\sqrt{x}$ (D) none of these

Q.5 If $y = f\left(\frac{2x-1}{x^2+1}\right)$ & $f'(x) = \sin x$ then $\frac{dy}{dx} =$

(A) $\frac{1+x-x^2}{\left(1+x^2\right)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$ (B) $\frac{2\left(1+x-x^2\right)}{\left(1+x^2\right)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$ (C) $\frac{1-x+x^2}{\left(1+x^2\right)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$ (D) none

Q.6 Let g is the inverse function of f & $f'(x) = \frac{x^{10}}{\left(1+x^2\right)}$. If g(2) = a then g′(2) is equal to

(A) $\frac{5}{2^{10}}$ (B) $\frac{1+a^2}{a^{10}}$ (C) $\frac{a^{10}}{1+a^2}$ (D) $\frac{1+a^{10}}{a^2}$

Q.7 If $\sin(xy) + \cos(xy) = 0$ then $\frac{dy}{dx} =$

(A) $\frac{y}{x}$ (B) $-\frac{y}{x}$ (C) $-\frac{x}{y}$ (D) $\frac{x}{y}$

Q.8 If $y = \sin^{-1}\frac{2x}{1+x^2}$ then $\left.\frac{dy}{dx}\right]_{x=-2}$ is :

(A) $\frac{2}{5}$ (B) $\frac{2}{\sqrt{5}}$ (C) $-\frac{2}{5}$ (D) none

Q.9 The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is :

(A) 4 (B) 1/4 (C) 1 (D) none

Q.10 If $y^2 = P(x)$, is a polynomial of degree 3, then $2\left(\frac{d}{dx}\right)\left(y^3 \cdot \frac{d^2y}{dx^2}\right)$ equals :

(A) $P'''(x) + P'(x)$ (B) $P''(x) \cdot P'''(x)$ (C) $P(x) \cdot P'''(x)$ (D) a constant

Q.11 Let f(x) be a quadratic expression which is positive for all real x . If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x, which one is correct .

(A) $g(x) < 0$ (B) $g(x) > 0$ (C) $g(x) = 0$ (D) $g(x) \geq 0$

Q.12 If $x^p \cdot y^q = (x + y)^{p+q}$ then $\frac{dy}{dx}$ is :

(A) independent of p but dependent on q (B) dependent on p but independent of q
(C) dependent on both p & q (D) independent of p & q both .

Q.13 Let $f(x) = \begin{cases} g(x) \cdot \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ where g(x) is an even function differentiable at x = 0, passing through the origin . Then $f'(0)$:(A) is equal to 1 (B) is equal to 0 (C) is equal to 2 (D) does not exist

Q.14 If $y = \frac{1}{1 + x^{n-m} + x^{p-m}} + \frac{1}{1 + x^{m-n} + x^{p-n}} + \frac{1}{1 + x^{m-p} + x^{n-p}}$ then $\frac{dy}{dx}$ at $e^{m^{n^p}}$ is equal to:

(A) e^{mnp} (B) $e^{mn/p}$ (C) $e^{np/m}$ (D) none

Q.15 $\lim_{x\to 0} \frac{\log_{\sin^2 x}\cos x}{\log_{\sin^2 \frac{x}{2}}\cos\frac{x}{2}}$ has the value equal to

(A) 1 (B) 2 (C) 4 (D) none of these

Q.16 If f is differentiable in (0, 6) & $f'(4) = 5$ then $\operatorname{Limit}_{x\to 2} \frac{f(4) - f(x^2)}{2 - x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

Q.17 Let $l = \lim_{x\to 0^+} x^m (\ln x)^n$ where m, n ∈ N then :

(A) l is independent of m and n (B) l is independent of m and depends on m
(C) l is independent of n and dependent on m (D) l is dependent on both m and n

Q.18 Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. Then $\operatorname{Limit}_{x\to 0} \frac{f'(x)}{x} =$

(A) 2 (B) −2 (C) −1 (D) 1

Q.19 Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right) =$

(A) 0 (B) −12 (C) 4 (D) 12

Q.20 People living at Mars, instead of the usual definition of derivative D f(x), define a new kind of derivative, D*f(x) by the formula

$D^*f(x) = \operatorname{Limit}_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h}$ where $f^2(x)$ means $[f(x)]^2$. If $f(x) = x \ln x$ then

$D^*f(x)\big|_{x=e}$ has the value

(A) e (B) 2e (C) 4e (D) none

Q.21 If $f(4) = g(4) = 2$; $f'(4) = 9$; $g'(4) = 6$ then $\operatorname{Limit}_{x\to 4} \frac{\sqrt{f(x)} - \sqrt{g(x)}}{\sqrt{x} - 2}$ is equal to :

(A) $3\sqrt{2}$ (B) $\frac{3}{\sqrt{2}}$ (C) 0 (D) none

Q.22 If f(x) is a differentiable function of x then $\operatorname{Limit}_{h\to 0} \frac{f(x+3h) - f(x-2h)}{h} =$

(A) $f'(x)$ (B) $5f'(x)$ (C) 0 (D) none

Q.23 If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\frac{e^x}{(1+e^x)^3}$ (C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{-1}{(1+e^x)^3}$

Q.24 If $x^2y + y^3 = 2$ then the value of $\frac{d^2y}{dx^2}$ at the point (1, 1) is :

(A) $-\frac{3}{4}$ (B) $-\frac{3}{8}$ (C) $-\frac{5}{12}$ (D) none

Q.25 If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$ then the value of $\underset{x \to a}{\text{Limit}} \frac{g(x) . f(a) - g(a) . f(x)}{x - a}$ is:

(A) -5 (B) 1/5 (C) 5 (D) none

Q.26 If f is twice differentiable such that $f''(x) = -f(x)$, $f'(x) = g(x)$

$$h'(x) = [f(x)]^2 + [g(x)]^2 \text{ and}$$
$$h(0) = 2,\ h(1) = 4$$

then the equation $y = h(x)$ represents :

(A) a curve of degree 2
(B) a curve passing through the origin
(C) a straight line with slope 2
(D) a straight line with y intercept equal to -2.

Q.27 The derivative of the function, $f(x)=\cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\} + \sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x + 3\sin x)\right\}$ w.r.t. $\sqrt{1 + x^2}$ at $x = \frac{3}{4}$ is :

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{10}{3}$ (D) 0

Q.28 Let $f(x)$ be a polynomial in x. Then the second derivative of $f(e^x)$, is :

(A) $f''(e^x) . e^x + f'(e^x)$
(B) $f''(e^x) . e^{2x} + f'(e^x) . e^{2x}$
(C) $f''(e^x) e^{2x}$
(D) $f''(e^x) . e^{2x} + f'(e^x) . e^x$

Q.29 The solution set of $f'(x) > g'(x)$, where $f(x) = \frac{1}{2}(5^{2x+1})$ & $g(x) = 5^x + 4x(\ln 5)$ is :

(A) $x > 1$ (B) $0 < x < 1$ (C) $x \le 0$ (D) $x > 0$

Q.30 If $y = \sin^{-1}\frac{x^2 - 1}{x^2 + 1} + \sec^{-1}\frac{x^2 + 1}{x^2 - 1}$, $|x| > 1$ then $\frac{dy}{dx}$ is equal to :

(A) $\frac{x}{x^4 - 1}$ (B) $\frac{x^2}{x^4 - 1}$ (C) 0 (D) 1

Q.31 If $y = \frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}\ldots\ldots \infty$ then $\frac{dy}{dx} =$

(A) $\frac{a}{ab + 2ay}$ (B) $\frac{b}{ab + 2by}$ (C) $\frac{a}{ab + 2by}$ (D) $\frac{b}{ab + 2ay}$

Q.32 Let f (x) be a polynomial function of second degree. If f (1) = f (–1) and a, b, c are in A.P., then f '(a), f '(b) and f '(c) are in

(A) G.P. (B) H.P. (C) A.G.P. (D) A.P.

Q.33 If $y = \sin mx$ then the value of $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ (where subscripts of y shows the order of derivatiive) is:

(A) independent of x but dependent on m
(B) dependent of x but independent of m
(C) dependent on both m & x
(D) independent of m & x .

Q.34 If $x^2 + y^2 = R^2$ $(R > 0)$ then $k = \frac{y''}{\sqrt{(1 + y'^2)^3}}$ where k in terms of R alone is equal to

(A) $-\frac{1}{R^2}$ (B) $-\frac{1}{R}$ (C) $\frac{2}{R}$ (D) $-\frac{2}{R^2}$

Q.35 If f & g are differentiable functions such that $g'(a) = 2$ & $g(a) = b$ and if fog is an identity function then $f'(b)$ has the value equal to :

(A) 2/3 (B) 1 (C) 0 (D) 1/2

Q.36 Given $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5a - x \sin a . \sin 2a - 5 \arcsin (a^2 - 8a + 17)$ then :

(A) $f(x)$ is not defined at $x = \sin 8$
(B) $f'(\sin 8) > 0$

(C) $f'(x)$ is not defined at $x = \sin 8$ (D) $f'(\sin 8) < 0$

Q.37 A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Then the value of $f'(4) =$
(A) 12 (B) 3 (C) 3/2 (D) cannot be determined

Q.38 Given : $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \sin 2a \,.\, \sin 6a + \sqrt{\ell n\left(2a - a^2\right)}$ then :
(A) $f(x)$ is not defined at $x = 1/2$ (B) $f'(1/2) < 0$
(C) $f'(x)$ is not defined at $x = 1/2$ (D) $f'(1/2) > 0$

Q.39 If $y = (A + Bx)\, e^{mx} + (m - 1)^{-2}\, e^x$ then $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y$ is equal to :
(A) e^x (B) e^{mx} (C) e^{-mx} (D) $e^{(1-m)x}$

Q.40 Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product *ab* is equal to
(A) 25 (B) 9 (C) – 15 (D) – 9

Q.41 Let h (x) be differentiable for all x and let $f(x) = (kx + e^x)\, h(x)$ where k is some constant. If $h(0) = 5$, $h'(0) = -2$ and $f'(0) = 18$ then the value of k is equal to
(A) 5 (B) 4 (C) 3 (D) 2.2

Q.42 Let $e^{f(x)} = \ln x$. If $g(x)$ is the inverse function of $f(x)$ then $g'(x)$ equals to :
(A) e^x (B) $e^x + x$ (C) $e^{(x + e^x)}$ (D) $e^{(x + \ln x)}$

Q.43 The equation $y^2e^{xy} = 9e^{-3}\cdot x^2$ defines y as a differentiable function of x. The value of $\frac{dy}{dx}$ for $x = -1$ and $y = 3$ is
(A) $-\frac{15}{2}$ (B) $-\frac{9}{5}$ (C) 3 (D) 15

Q.44 Let $f(x) = \left(x^x\right)^x$ and $g(x) = x^{\left(x^x\right)}$ then :
(A) $f'(1) = 1$ and $g'(1) = 2$ (B) $f'(1) = 2$ and $g'(1) = 1$
(C) $f'(1) = 1$ and $g'(1) = 0$ (D) $f'(1) = 1$ and $g'(1) = 1$

Q.45 The function $f(x) = e^x + x$, being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\ln 2)$ is
(A) $\frac{1}{\ell n 2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) none

Q.46 If $f(x) = \dfrac{\log_{\sin|x|}\cos^3 x}{\log_{\sin|3x|}\cos^3\left(\frac{x}{2}\right)}$ for $|x| < \frac{\pi}{3}$ $x \neq 0$
$= 4$ for $x = 0$
then, the number of points of discontinuity of f in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ is
(A) 0 (B) 3 (C) 2 (D) 4

Q.47 If $y = \dfrac{(a-x)\sqrt{a-x} - (b-x)\sqrt{x-b}}{\sqrt{a-x} + \sqrt{x-b}}$ then $\frac{dy}{dx}$ wherever it is defined is equal to :
(A) $\dfrac{x + (a+b)}{\sqrt{(a-x)(x-b)}}$ (B) $\dfrac{2x - (a+b)}{2\sqrt{(a-x)(x-b)}}$ (C) $-\dfrac{(a+b)}{2\sqrt{(a-x)(x-b)}}$ (D) $\dfrac{2x + (a+b)}{2\sqrt{(a-x)(x-b)}}$

Q.48 If y is a function of x then $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$. If x is a function of y then the equation becomes :
(A) $\frac{d^2x}{dy^2} + x\frac{dx}{dy} = 0$ (B) $\frac{d^2x}{dy^2} + y\left(\frac{dx}{dy}\right)^3 = 0$ (C) $\frac{d^2x}{dy^2} - y\left(\frac{dx}{dy}\right)^2 = 0$ (D) $\frac{d^2x}{dy^2} - x\left(\frac{dx}{dy}\right)^2 = 0$

Q.49 A function $f(x)$ satisfies the condition, $f(x) = f'(x) + f''(x) + f'''(x) + \ldots\ldots \infty$ where $f(x)$ is a differentiable function indefinitely and dash denotes the order of derivative . If $f(0) = 1$, then $f(x)$ is :
(A) $e^{x/2}$ (B) e^x (C) e^{2x} (D) e^{4x}

Q.50 If $y = \dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$, then $\frac{dy}{dx} =$
(A) 2 sinx + cosx (B) –2sinx (C) cos2x (D) sin2x

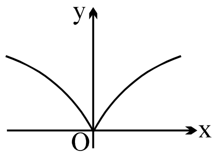
Q.51 If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = K$ then the value of K is equal to
(A) 1 (B) –1 (C) 2 (D) 0

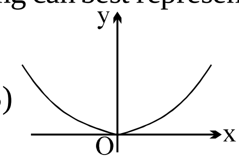
Q.52 If $f(x) = 2\sin^{-1}\sqrt{1-x} + \sin^{-1}\left(2\sqrt{x(1-x)}\right)$ where $x \in \left(0, \frac{1}{2}\right)$ then $f'(x)$ has the value equal to

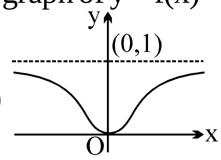
(A) $\frac{2}{\sqrt{x(1-x)}}$ (B) zero (C) $-\frac{2}{\sqrt{x(1-x)}}$ (D) π

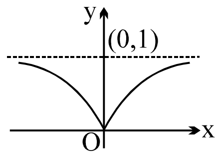
Q.53 Let $y = f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then which of the following can best represent the graph of $y = f(x)$?

(A) 

(B) 

(C) 

(D) 

Q.54 Diffrential coefficient of $\left(x^{\frac{\ell+m}{m-n}}\right)^{\frac{1}{n-\ell}} \cdot \left(x^{\frac{m+n}{n-\ell}}\right)^{\frac{1}{\ell-m}} \cdot \left(x^{\frac{n+\ell}{\ell-m}}\right)^{\frac{1}{m-n}}$ w.r.t. x is

(A) 1 (B) 0 (C) – 1 (D) $x^{\ell mn}$

Q.55 Let $f(x)$ be diffrentiable at $x = h$ then $\lim_{x\to h} \frac{(x+h)\, f(x) - 2h\, f(h)}{x-h}$ is equal to

(A) $f(h) + 2hf'(h)$ (B) $2f(h) + hf'(h)$ (C) $hf(h) + 2f'(h)$ (D) $hf(h) - 2f'(h)$

Q.56 If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\frac{d^3y}{dx^3}$ equals

(A) $24a^2(at + b)$ (B) $24a(ax + b)^2$ (C) $24a(at + b)^2$ (D) $24a^2(ax + b)$

Q.57 $\lim_{x\to 0^+} \frac{1}{x\sqrt{x}}\left(a \arctan\frac{\sqrt{x}}{a} - b \arctan\frac{\sqrt{x}}{b}\right)$ has the value equal to

(A) $\frac{a-b}{3}$ (B) 0 (C) $\frac{(a^2-b^2)}{6a^2b^2}$ (D) $\frac{a^2-b^2}{3a^2b^2}$

Q.58 Let $f(x)$ be defined for all $x > 0$ & be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y & $f(e) = 1$. Then :

(A) $f(x)$ is bounded (B) $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$ (C) $x.f(x) \to 1$ as $x \to 0$ (D) $f(x) = \ln x$

Q.59 Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x)$ at $x = 1$, has the value equal to

(A) 19 (B) 9 (C) 17 (D) 14

Q.60 If $y = \frac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1}$ and $\frac{dy}{dx} = ax + b$ then the value of $a + b$ is equal to

(A) $\cot\frac{5\pi}{8}$ (B) $\cot\frac{5\pi}{12}$ (C) $\tan\frac{5\pi}{12}$ (D) $\tan\frac{5\pi}{8}$

Q.61 Suppose that $h(x) = f(x)\cdot g(x)$ and $F(x) = f\left(g(x)\right)$, where $f(2) = 3$; $g(2) = 5$; $g'(2) = 4$; $f'(2) = -2$ and $f'(5) = 11$, then

(A) $F'(2) = 11\, h'(2)$ (B) $F'(2) = 22h'(2)$ (C) $F'(2) = 44\, h'(2)$ (D) none

Q.62 Let $f(x) = x^3 + 8x + 3$

which one of the properties of the derivative enables you to conclude that $f(x)$ has an inverse?

(A) $f'(x)$ is a polynomial of even degree. (B) $f'(x)$ is self inverse.

(C) domain of $f'(x)$ is the range of $f'(x)$. (D) $f'(x)$ is always positive.

Q.63 Which one of the following statements is NOT CORRECT ?

(A) The derivative of a diffrentiable periodic function is a periodic function with the same period.

(B) If f (x) and g (x) both are defined on the entire number line and are aperiodic then the function $F(x) = f(x) \cdot g(x)$ can not be periodic.

(C) Derivative of an even differentiable function is an odd function and derivative of an odd differentiable function is an even function.

(D) Every function f (x) can be represented as the sum of an even and an odd function

<u>Select the correct alternatives</u> : (More than one are correct)

Q.64 If $y = \tan x \tan 2x \tan 3x$ then $\frac{dy}{dx}$ has the value equal to :

(A) $3\sec^2 3x\ \tan x\ \tan 2x + \sec^2 x\ \tan 2x\ \tan 3x + 2\sec^2 2x\ \tan 3x\ \tan x$
(B) $2y(\text{cosec}\,2x + 2\,\text{cosec}\,4x + 3\,\text{cosec}\,6x)$
(C) $3\sec^2 3x - 2\sec^2 2x - \sec^2 x$ (D) $\sec^2 x + 2\sec^2 2x + 3\sec^2 3x$

Q.65 If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ then $\frac{dy}{dx}$ equals
(A) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$ (B) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2x}$ (C) $\frac{1}{2\sqrt{x}}\sqrt{y^2-4}$ (D) $\frac{1}{2\sqrt{x}}\sqrt{y^2+4}$

Q.66 If $y = x^{x^2}$ then $\frac{dy}{dx}$ = (A) $2\,ln\,x\,.\,x^{x^2}$ (B) $(2\,ln\,x + 1).\,x^{x^2}$ (C) $(2\,ln\,x+1).\,x^{x^2+1}$ (D) $x^{x^2+1}.\,ln\,ex^2$

Q.67 Let $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$ then $\frac{dy}{dx}$ =
(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x+2y}$ (C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

Q.68 If $2^x + 2^y = 2^{x+y}$ then $\frac{dy}{dx}$ has the value equal to :
(A) $-\frac{2^y}{2^x}$ (B) $\frac{1}{1-2^x}$ (C) $1-2^y$ (D) $\frac{2^x\left(1-2^y\right)}{2^y\left(2^x-1\right)}$

Q.69 The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation :
(A) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ (B) $\frac{d^2u}{dx^2} = 2v$ (C) $\frac{d^2v}{dx^2} = -2u$ (D) none of these

Q.70 Let $f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1}-1}.\,x$ then :
(A) $f'(10) = 1$ (B) $f'(3/2) = -1$ (C) domain of $f(x)$ is $x \ge 1$ (D) none

Q.71 Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,
$f(0) = \frac{2}{g(0)}$, $f'(0) = 2g'(0) = 4g(0)$, $g''(0) = 5f''(0) = 6f(0) = 3$ then :
(A) if $h(x) = \frac{f(x)}{g(x)}$ then $h'(0) = \frac{15}{4}$ (B) if $k(x) = f(x)\,.\,g(x)\sin x$ then $k'(0) = 2$
(C) $\underset{x\to 0}{\text{Limit}}\ \frac{g'(x)}{f'(x)} = \frac{1}{2}$ (D) none

Q.72 If $y = x^{(\ell n x)^{\ell n(\ell n x)}}$, then $\frac{dy}{dx}$ is equal to :
(A) $\frac{y}{x}\left(\ell n x^{\ell n x - 1} + 2\ell n x\ \ell n\left(\ell n x\right)\right)$ (B) $\frac{y}{x}(ln\,x)^{ln\,(ln\,x)}(2\,ln\,(ln\,x)+1)$
(C) $\frac{y}{x\,\ell n x}\left((ln\,x)^2 + 2\,ln\,(ln\,x)\right)$ (D) $\frac{y\,\ell n y}{x\,\ell n x}(2\,ln\,(ln\,x)+1)$

ANSWER KEY

Q.1	A	Q.2	C	Q.3	B	Q.4	D	Q.5	B
Q.6	B	Q.7	B	Q.8	C	Q.9	A	Q.10	C
Q.11	B	Q.12	D	Q.13	B	Q.14	D	Q.15	C
Q.16	D	Q.17	A	Q.18	B	Q.19	C	Q.20	C
Q.21	A	Q.22	B	Q.23	B	Q.24	B	Q.25	C
Q.26	C	Q.27	C	Q.28	D	Q.29	D	Q.30	C
Q.31	D	Q.32	D	Q.33	D	Q.34	B	Q.35	D
Q.36	D	Q.37	B	Q.38	D	Q.39	A	Q.40	C
Q.41	C	Q.42	C	Q.43	D	Q.44	D	Q.45	B
Q.46	C	Q.47	B	Q.48	C	Q.49	A	Q.50	B
Q.51	D	Q.52	B	Q.53	C	Q.54	B	Q.55	A
Q.56	D	Q.57	D	Q.58	D	Q.59	A	Q.60	B
Q.61	B	Q.62	D	Q.63	B				
Q.64	A,B,C	Q.65	A,C	Q.66	C,D	Q.67	A,C,D		
Q.68	A,B,C,D	Q.69	A,B,C	Q.70	A,B	Q.71	A,B,C	Q.72	B,D

EXERCISE -1

Part : (A) Only one correct option

1. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$ then $\frac{dy}{dx}$ at x = 1 is

(A) 2 (B) 1 (C) – 2 (D) –1

2. If $y = x^{x^2}$ then $\frac{dy}{dx}$ =

(A) $2\,\ell n\, x.\ x^{x^2}$ (B) $(2\,\ell n\, x + 1).\ x^{x^2}$ (C) $(2\,\ell n\, x + 1).\ x^{x^2+1}$ (D) $x^{x^2+1}.\ \ell n\, ex^2$

3. If $f(x) = e^{\tan^{-1}\left(\sin \frac{x}{2}\right)}$, then $f'(0)$.

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 1 (D) –1

4. If $y = \cfrac{x}{a + \cfrac{x}{b + \cfrac{x}{a + ...}}}$ then $\frac{dy}{dx}$ =

(A) $\frac{a}{ab + 2ay}$ (B) $\frac{b}{ab + 2by}$ (C) $\frac{a}{ab + 2by}$ (D) $\frac{b}{ab + 2ay}$

5. Let $f(x) = \sin x$; $g(x) = x^2$ & $h(x) = \log_e x$ & $F(x) = h[g(f(x))]$ then $\frac{d^2F}{dx^2}$ is equal to:

(A) $2\,\text{cosec}^3 x$ (B) $2\cot(x^2) - 4x^2\,\text{cosec}^2(x^2)$ (C) $2x\cot x^2$ (D) $-2\,\text{cosec}^2 x$

6. If $y = (1 + x)(1 + x^2)(1 + x^4) .....(1 + x^{2n})$, then $\frac{dy}{dx}$ at x = 0 is

(A) –1 (B) 1 (C) 0 (D) 2^n

7. If $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$ and $\frac{dy}{dx} = \frac{1}{2\sqrt{x(1-x)}}$ + p, then p =

(A) 0 (B) $\frac{1}{\sqrt{1-x}}$ (C) $\sin^{-1}\sqrt{x}$ (D) $\frac{1}{\sqrt{1-x^2}}$

8. If $\sqrt{x^2 + y^2} = e^t$ where $t = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$ then $\frac{dy}{dx}$:

(A) $\frac{x-y}{x+y}$ (B) $\frac{x+y}{x-y}$ (C) $\frac{2x+y}{x-y}$ (D) $\frac{x-y}{2x+y}$

9. If $y = \sin^{-1}\frac{x^2-1}{x^2+1} + \sec^{-1}\frac{x^2+1}{x^2-1}$, $|x| > 1$ then $\frac{dy}{dx}$ is equal to:

(A) $\frac{x}{x^4-1}$ (B) $\frac{x^2}{x^4-1}$ (C) 0 (D) 1

10. The differential coefficient of $\sin^{-1}\frac{t}{\sqrt{1+t^2}}$ w.r.t. $\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ is:

(A) 1 (B) t (C) $\frac{1}{\sqrt{1+t^2}}$ (D) none

11. Differentiation of $\left(\frac{\tan^{-1} x}{1 + \tan^{-1} x}\right)$ w.r.t. $\tan^{-1} x$ is:

(A) $\left(\frac{1}{1+\tan^{-1} x}\right)$ (B) – 1 (C) $\frac{1}{\left(1+\tan^{-1} x\right)^2}$ (D) $\frac{-1}{(1+\tan^{-1} x)^2}$

12. Let f(x) be a polynomial in x. Then the second derivative of $f(e^x)$, is:

(A) $f''(e^x).\ e^x + f'(e^x)$ (B) $f''(e^x).\ e^{2x} + f'(e^x).\ e^{2x}$ (C) $f''(e^x)\ e^{2x}$ (D) $f''(e^x).\ e^{2x} + f'(e^x).\ e^x$

13. If f(x), g(x), h(x) are polynomials in x of degree 2 and $F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$, then $F'(x)$ is equal to

(A) 1 (B) 0 (C) –1 (D) f(x) . g(x) . h(x)

14. If y = sin mx then the value of $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ (where settings of y shows the order of derivative) is:

(A) independent of x but dependent on m (B) dependent of x but independent of m
(C) dependent on both m & x (D) independent of m & x.

15. If $f'(5) = 7$ then $\lim_{t\to 0} \frac{f(5+t) - f(5-t)}{2t} =$

(A) 0 (B) 3.5 (C) 7 (D) 14

16. Let $e^{f(x)} = \ln x$. If g(x) is the inverse function of f(x) then g′(x) equals to:

(A) e^x (B) $e^x + x$ (C) e^{x+e^x} (D) $e^{x + \ln x}$

17. If u = ax + b then $\frac{d^n}{dx^n}$ [f(ax + b)] is equal to:

(A) $\frac{d^n}{du^n}$ [f(u)] (B) a $\frac{d^n}{du^n}$ [f(u)] (C) $a^n \frac{d^n}{du^n}$ [f(u)] (D) $a^{-n} \frac{d^n}{dx^n}$ [f(u)]

18. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ & $f'(x) = \sin x$ then $\frac{dy}{dx} =$

(A) $\frac{1+x-x^2}{(1+x^2)^2} \sin\left(\frac{2x-1}{x^2+1}\right)$ (B) $\frac{2(1+x-x^2)}{(1+x^2)^2} \sin\left(\frac{2x-1}{x^2+1}\right)$

(C) $\frac{1-x+x^2}{(1+x^2)^2} \sin\left(\frac{2x-1}{x^2+1}\right)$ (D) none

19. If $y^2 = P(x)$, is a polynomial of degree 3, then $2\left(\frac{d}{dx}\right)\left(y^3 . \frac{d^2y}{dx^2}\right)$ equals:

(A) P′′′(x) + P′(x) (B) P′′(x). P′′′(x) (C) P(x). P′′′(x) (D) a constant

Part : (B) May have more than one options correct

20. Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,

$f(0) = \frac{2}{g(0)}$, f′(0) = 2 g′(0) = 4g(0), g′′(0) = 5 f′′(0) = 6 f(0) = 3 then:

(A) if $h(x) = \frac{f(x)}{g(x)}$ then $h'(0) = \frac{15}{4}$ (B) if k(x) = f(x). g(x) sin x then k′(0) = 2

(C) $\lim_{x\to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}$ (D) none

21. If $f_n(x) = e^{f_{n-1}(x)}$ for all n ∈ N and $f_o(x) = x$, then $\frac{d}{dx}\{f_n(x)\}$ is equal to:

(A) $f_n(x). \frac{d}{dx}\{f_{n-1}(x)\}$ (B) $f_n(x). f_{n-1}(x)$

(C) $f_n(x). f_{n-1}(x)........ f_2(x). f_1(x)$ (D) none of these

22. If f is twice differentiable such that f′′(x) = –f(x) and f′(x) = g(x). If h(x) is a twice differentiable function such that h′(x) = [f(x)]² + [g(x)]². If h(0) = 2, h(1) = 4, then the equation y = h(x) represents:

(A) a curve of degree 2 (B) a curve passing through the origin
(C) a straight line with slope 2 (D) a straight line with y intercept equal to 2.

23. Given $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5a - x \sin a . \sin 2a - 5 \sin^{-1}(a^2 - 8a + 17)$ then:

(A) $f'(x) = -x^2 + 2x \sin 6 - \sin 4 \sin 8$ (B) f′(sin 8) > 0
(C) f′(x) is not defined at x = sin 8 (D) f′(sin 8) < 0

24. If $f(x) = x^3 + x^2 f'(1) + x f''(3)$ for all x ∈ R then

(A) f(0) + f(2) = f(1) (B) f(0) + f(3) = 0 (C) f(1) + f(3) = f(2) (D) none of these

25. If f(x) = (ax + b) sin x + (cx + d) cos x, then the values of a, b, c and d such that f′(x) = x cos x for all x are

(A) a = d = 1 (B) b = 0 (C) c = 0 (D) b = c

EXERCISE -2

1. If $y = A e^{-kt} \cos(pt + c)$ then prove that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2 y = 0$, where $n^2 = p^2 + k^2$.

Evaluate the following limits using L′ hospitale rule as otherwise

2. $\lim_{x\to 0} \log_{\tan^2 x} (\tan^2 2x)$

3. If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then $f'(x) = \lambda . \begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$. Find the value of λ.

4. If $x = a\,t^3$ & $y = b\,t^2$, where t is a parameter, then prove that $\frac{d^3y}{dx^3} = \frac{8.b}{27a^3.t^7}$

5. If sin y = x sin (a + y), show that $\frac{dy}{dx} = \frac{\sin a}{1-2x\cos a+x^2}$.

6. If F(x) = f(x). g(x) & f ′(x). g ′(x) = c, prove that $\frac{F''}{F} = \frac{f''}{f} + \frac{g''}{g} + \frac{2c}{fg}$ & $\frac{F'''}{F} = \frac{f'''}{f} + \frac{g'''}{g}$.

7. If α be a repeated root of a quadratic equation f(x) = 0 & A(x), B(x), C(x) be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where dash denotes the derivative.

8. Show that $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ can be reduced to the form $R^{2/3} = \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}} + \frac{1}{\left(\frac{d^2x}{dy^2}\right)^{\frac{2}{3}}}$.

Also show that, if $x = a \sin 2\theta\,(1 + \cos 2\theta)$ & $y = a \cos 2\theta\,(1 - \cos 2\theta)$ then the value of R equals to $4\,a \cos 3\theta$.

9. Differentiate the following functions with respect to x.

(i) $x^2.\ \ell n\ x.\ e^x$ (i) $\frac{\sin x - x\cos x}{x\sin x + \cos x}$ (iii) $\tan\left(\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Exercise # 1

1. A **2.** C **3.** A **4.** D **5.** D **6.** B **7.** D **8.** B **9.** C **10.** A **11.** C
12. D **13.** B **14.** D **15.** C **16.** C **17.** C **18.** B **19.** C **20.** ABC
21. AC **22.** CD **23.** AD **24.** ABC **25.** ABC

Exercise # 2

2. 1 **3.** 3

9. (i) $e^x\,x\,(2\,\ell n\,x + 1 + x\,\ell n\,x)$ (ii) $\frac{x^2}{(x\sin x+\cos x)^2}$ (iii) $\frac{1}{2}\sec^2\frac{x}{2}$

