# SIMPLE HARMONIC MOTION

## 1. PERIODIC MOTION

When a body or a moving particle repeats its motion along a definite path after regular intervals of time, its motion is said to be **Periodic Motion** and interval of time is called **time period** or harmonic motion period (T). The path of periodic motion may be linear, circular, elliptical or any other curve. For example, rotaion of earth about the sun.

## 2. OSCILLATORY MOTION

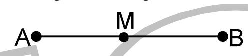
'To and Fro' type of motion is called an **Oscillatory Motion**. It need not be periodic and need not have fixed extreme positions. For example, motion of pendulum of a wall clock.
The oscillatory motions in which energy is conserved are also periodic.
The force / torque (directed towards equilibrium point) acting in oscillatory motion is called restoring force / torque.
**Damped oscillations** are those in which energy is consumed due to some resistive forces and hence total mechanical energy decreases.

## 3. SIMPLE HARMONIC MOTION

If the restoring force/ torque acting on the body in oscillatory motion is directly proportional to the displacement of body/particle and is always directed towards equilibrium position then the motion is called simple Harmonic Motion (SHM). It is the simplest (easy to analyse) form of oscillatory motion.

### 3.1 TYPES OF SHM

**(a) Linear SHM :** When a particle moves to and fro about an equilibrium point, along a straight line. A and B are extreme positions. M is mean position. AM = MB = Amplitude




**(b) Angular SHM :** When body/particle is free to rotate about a given axis executing angular oscillations.



### 3.2 EQUATION OF SIMPLE HARMONIC MOTION (SHM) :

The necessary and sufficient condition for SHM is

$$F = -kx$$

where k = positive constant for a SHM = Force constant
x = displacement from mean position.

or $$m\frac{d^2x}{dt^2} = -kx$$

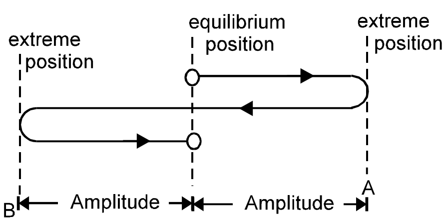
$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{[differential equation of SHM]}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

It's solution is $x = A \sin(\omega t + \phi)$

### 3.4 CHARACTERISTICS OF SHM

**Note :** In the figure shown, path of the particle is on a straight line.
**(a) Displacement -** It is defined as the distance of the particle from the mean position at that instant.
Displacement in SHM at time t is given by $x = A \sin(\omega t + \phi)$
**(b) Amplitude -** It is the maximum value of displacement of the particle from its equilibrium position.




Amplitude = $\frac{1}{2}$ [distance between extreme points/position]
It depends on energy of the system.

**(c) Angular Frequency (ω) :** $\omega = \frac{2\pi}{T} = 2\pi f$ and its units is rad/sec.

**(d) Frequency (f) :** Number of oscillations completed in unit time interval is called frequency of oscillations, $f = \frac{1}{T} = \frac{\omega}{2\pi}$, its units is $sec^{-1}$ or Hz.

**(e) Time period (T) :** Smallest time interval after which the oscillatory motion gets repeated is called time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

**Ex. 1** For a particle performing SHM, equation of motion is given as $\frac{d^2x}{dt^2} + 4x = 0$. Find the time period.

**Sol.** $\frac{d^2x}{dt^2} = -4x$ $\quad\quad \omega^2 = 4 \quad\quad \omega = 2$

Time period; $\quad T = \frac{2\pi}{\omega} = \pi$

**(f) Phase :** The physical quantity which represents the state of motion of particle (eg. its position and direction of motion at any instant).
The argument $(\omega t + \phi)$ of sinusoidal function is called instantaneous phase of the motion.

**(g) Phase constant ($\phi$) :** Constant $\phi$ in equation of SHM is called phase constant or initial phase.
It depends on initial position and direction of velocity.

**(h) Velocity(v) :** It is the rate of change of particle's displacemnet w.r.t time at that instant.
Let the displacemnet from mean position is given by

$$x = A \sin(\omega t + \phi)$$

Velocity, $\quad v = \frac{dx}{dt} = \frac{d}{dt}[A\sin(\omega t + \phi)] \quad\quad v = A\omega\cos(\omega t + \phi)$

or, $\quad v = \omega\sqrt{A^2 - x^2}$

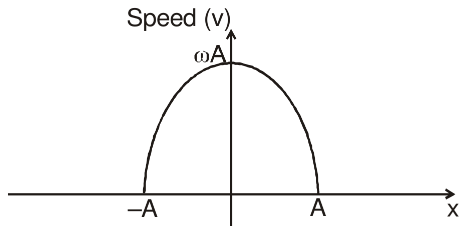
At mean position (x = 0), velocity is maximum.

$$v_{max} = \omega A$$

At extreme position (x = A), velocity is minimum.

$$v_{min} = \text{zero}$$

***GRAPH OF SPEED (v) VS DISPLACEMENT (x):***




$v = \omega\sqrt{A^2 - x^2} \quad\quad v^2 = \omega^2(A^2 - x^2)$

$v^2 + \omega^2x^2 = \omega^2A^2 \quad\quad \frac{v^2}{\omega^2A^2} + \frac{x^2}{A^2} = 1$

**GRAPH WOULD BE AN ELLIPSE**

**(i) Acceleration :** It is the rate of change of particle's velocity w.r.t. time at that instant.

Acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}[A\omega\cos(\omega t + \phi)]$

$$a = -\omega^2 A \sin(\omega t + \phi)$$

$$a = -\omega^2 x$$

**NOTE:** Negative sign shows that acceleration is always directed towards the mean postion.
At mean position (x = 0), acceleration is minimum.

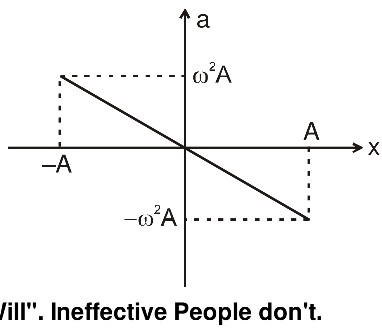
$$a_{min} = \text{zero}$$

At extreme position (x = A), acceleration is maximum.

$$a_{max} = \omega^2 A$$

***GRAPH OF ACCELERATION (A) VS DISPLACEMENT (x)***

$$a = -\omega^2 x$$



**Ex. 2** The equation of particle executing simple harmonic motion is $x = (5\text{ m})\sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at t = 1 s.

**Sol.** Comparing with equation $x = A \sin(\omega t + \delta)$, we see that
the amplitude = 5 m,

and time period = $\frac{2\pi}{\omega} = \frac{2\pi}{\pi\, s^{-1}} = 2s.$

The maximum speed = $A\omega = 5\text{ m} \times \pi\, s^{-1} = 5\pi$ m/s.

The velocity at time t = $\frac{dx}{dt} = A\omega\cos(\omega t + \delta)$

At t = 1 s, $v = (5\text{ m})(\pi\, s^{-1})\cos\left(\pi + \frac{\pi}{5}\right) = -\frac{5\pi}{2}$ m/s.

**Ex. 3** A particle executing simple harmonic motion has angular frequency 6.28 $s^{-1}$ and amplitude 10 cm. Find (a) the time period, (b) the maximum speed, (c) the maximum acceleration, (d) the speed when the displacement is 6 cm from the mean position, (e) the speed at t = 1/6 s assuming that the motion starts from rest at t = 0.

**Sol.** (a) Time period = $\frac{2\pi}{\omega} = \frac{2\pi}{6.28}$ s = 1 s.

(b) Maximum speed = $A\omega = (0.1\text{ m})(6.28\, s^{-1})$
$= 0.628$ m/s.

(c) Maximum acceleration = $A\omega^2$
$= (0.1\text{ m})(6.28\, s^{-1})^2$
$= 4\text{ m/s}^2$.

(d) $v = \omega\sqrt{A^2 - x^2} = (6.28\, s^{-1})\sqrt{(10\,\text{cm})^2 - (6\text{ cm})^2}$
$= 50.2$ cm/s.

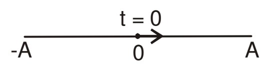
(e) At t = 0, the velocity is zero i.e., the particle is at an extreme. The equation for displacement may be written as
$x = A\cos\omega t.$

The velocity is $v = -A\omega\sin\omega t$.

At $t = \frac{1}{6}$s, $v = -(0.1\text{ m})(6.28\, s^{-1})\sin\left(\frac{6.28}{6}\right)$

$= (-0.628\text{ m/s})\sin\frac{\pi}{3}$ = 54.4 cm/s.

**Ex. 4** A particle starts from mean position and moves towards positive extreme as shown. Find the equation of the SHM. Amplitude of SHM is A.




**Sol.** General equation of SHM can be written as $x = A\sin(\omega t + \phi)$

At t = 0, x = 0

$\therefore$ $0 = A\sin\phi$

$\therefore$ $\phi = 0, \pi$ $\phi \in [0, 2\pi)$

Also; at t = 0, v = +ve

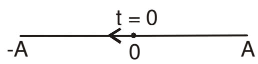
$\therefore$ $A\omega\cos\phi$ = +ve

or, $\phi = 0$

Hence, if the particle is at mean position at t = 0 and is moving towards +ve extreme, then the equation of SHM is given by $x = A\sin\omega t$

**Similarly**

for

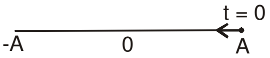


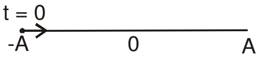

$\phi = \pi$

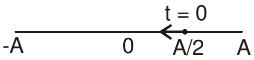
$\therefore$ equation of SHM is $x = A\sin(\omega t + \pi)$ or, $x = -A\sin\omega t$

**NOTE: If mean position is not at the origin, then we can replace x by $x - x_0$ and the eqn. becomes $x - x_0 = -A\sin\omega t$, where $x_0$ is the position co-ordinate of the mean position.**

**Q. 1** Write the equation of SHM for the situations shown below:

**(a)** 


**(b)** 


**(c)** 


**Ans.** (a) $x = A \cos\omega t$ ; (b) $x = -A \cos\omega t$ ; (c) $x = A \sin(\omega t + 150^\circ)$

**Ex. 5** A particle is performing SHM of amplitude "A" and time period "T". Find the time taken by the particle to go from 0 to A/2.

**Sol.** Let equation of SHM be $x = A \sin \omega t$

when $x = 0$ , $t = 0$

when $x = A/2$ ; $A/2 = A \sin \omega t$

or $\sin \omega t = 1/2$ $\omega t = \pi/6$

$$\frac{2\pi}{T} t = \pi/6 \qquad t = T/12$$

Hence , time taken is T/12, where T is time period of SHM.

**Ex. 6** A particle of mass 2 kg is moving on a straight line under the action force F = (8 – 2x) N. It is released at rest from x = 6 m.

(a) Is the particle moving simple harmonically.

(b) Find the equilibrium position of the particle.

(c) Write the equation of motion of the particle.

(d) Find the time period of SHM.

**Sol.** $F = 8 - 2x$ or $F = -2(x - 4)$

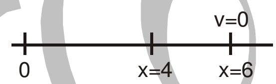
for equilibrium position F = 0

$\Rightarrow$ x = 4 is equilibrium position

Hence the motion of particle is SHM with force constant 2 and equilibrium position x = 4.

(a) Yes, motion is SHM.

(b) Equilibrium position is x = 4




(c) At x = 6 m, particle is at rest i.e. it is one of the extreme position

Hence amplitude is A = 2 m and initially particle is at the extreme position.

$\therefore$ Equation of SHM can be written as

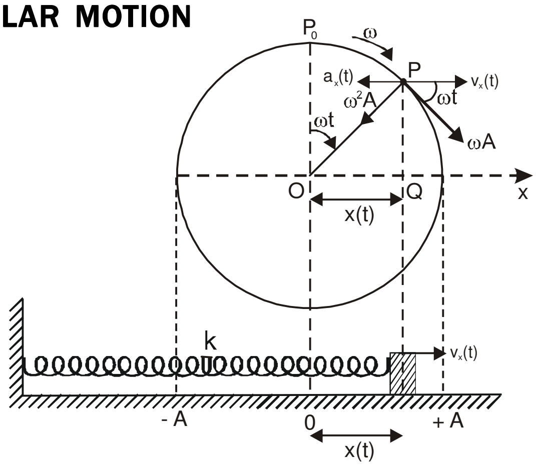
$x - 4 = 2 \cos \omega t$,

where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$

i.e. $x = 4 + 2 \cos t$

(d) Time period, $T = \frac{2\pi}{\omega} = 2\pi$ sec.

## 4. SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION




Consider a particle moving on a circle of radius A with a constant angular speed $\omega$ as shown in figure.

Suppose the particle is on the top of the circle (Y-axis) at t = 0. The radius OP make an angle $\theta = \omega t$ with the Y-axis at time t. Drop a perpendicular PQ on X-axis. The components of position vector, velocity vector and acceleration vector at time t on the X-axis are

$x(t) = A \sin \omega t$

$v_x(t) = A\omega \cos \omega t$

$a_x(t) = -\omega^2 A \sin \omega t$

Above equations show that the foot of perpendicular Q executes a simple harmonic motion on the X-axis. The amplitude is A and angular frequency is $\omega$. Similarly the foot of perpendicular on Y-axis will also executes SHM of amplitude A and angular frequency $\omega$ [$y(t) = A \cos \omega t$]. The phases of the two simple harmonic motions differ by $\pi/2$.

## 5. GRAPHICAL REPRESENTATION OF DISPLACEMENT, VELOCITY & ACCELERATION IN SHM

Displacement, $x = A \sin \omega t$

Velocity, $v = A\omega \cos \omega t = A\omega \sin (\omega t + \frac{\pi}{2})$

or $v = \omega \sqrt{A^2 - x^2}$

Acceleration, $a = -\omega^2 A \sin\omega t = \omega^2 A \sin (\omega t + \pi)$

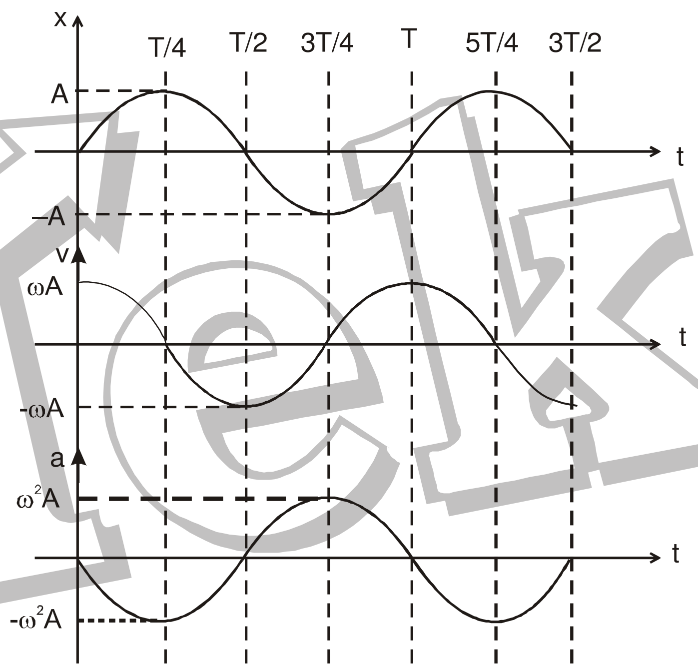
or $a = -\omega^2 x$

**Note :** $v = \omega \sqrt{A^2 - x^2}$

$a = -\omega^2 x$

These relations are true for any equation of x.

| time, t | 0 | T/4 | T/2 | 3T/4 | T |
|---|---|---|---|---|---|
| **displacement, x** | 0 | A | 0 | – A | 0 |
| **Velocity, v** | $A\omega$ | 0 | $-A\omega$ | 0 | $A\omega$ |
| **acceleration, a** | 0 | $-\omega^2 A$ | 0 | $\omega^2 A$ | 0 |




**1.** All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.

**2.** The velocity amplitude is $\omega$ times the displacement amplitude ($v_{max} = \omega A$).

**3.** The acceleration amplitude is $\omega^2$ times the displacement amplitude ($a_{max} = \omega^2 A$).

**4.** In SHM, the velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$.

**5.** In SHM, the acceleration is ahead of velocity by a phase angle of $\frac{\pi}{2}$.

## 6. ENERGY OF SHM

### 6.1 Kinetic Energy (KE)

$\frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} k (A^2 - x^2)$ (as a function of x)

$= \frac{1}{2}$ m $A^2\omega^2 \cos^2(\omega t + \theta) = \frac{1}{2}$ $KA^2 \cos^2(\omega t + \theta)$ (as a function of t)

$KE_{max} = \frac{1}{2}kA^2$ ; $\langle KE \rangle_{0-T} = \frac{1}{4}kA^2$ ; $\langle KE \rangle_{0-A} = \frac{1}{3}kA^2$

Frequency of KE = 2 (frequency of SHM)

**6.2 Potential Energy (PE)**

$\frac{1}{2}$ $Kx^2$ (as a function of x) $= \frac{1}{2}$ $kA^2 \sin^2(\omega t + \theta)$ (as a function of time)

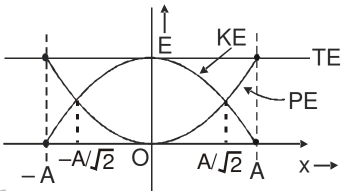
**6.3 Total Mechanical Energy (TME)**

Total mechanical energy = Kinetic energy + Potential energy

$$= \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}Kx^2 = \frac{1}{2}KA^2$$

Hence total mechanical energy is constant in SHM.

**6.4 Graphical Variation of energy of SHM.**




**Ex. 7** A particle of mass 0.50 kg executes a simple harmonic motion under a force F = – (50 N/m)x. If it crosses the centre of oscillation with a speed of 10 m/s, find the amplitude of the motion.

**Sol.** The kinetic energy of the particle when it is at the centre of oscillation is

$E = \frac{1}{2}mv^2 = \frac{1}{2}(0.50 \text{ kg})(10 \text{ m/s})^2$

= 25 J.

The potential energy is zero here. At the maximum displacement x = A, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2}kA^2$. As there is no loss of energy,

$\frac{1}{2}kA^2 = 25$ J .............(i)

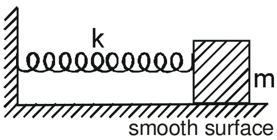
The force on the particle is given by F = – (50 N/m)x.

Thus, the spring constant is k = 50 N/m.

Equation (i) gives $\frac{1}{2}(50 \text{ N/m})A^2 = 25$ J or, A = 1 m.

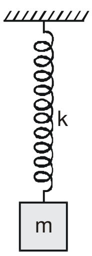
## 7. SPRING-MASS SYSTEM

**(1)**



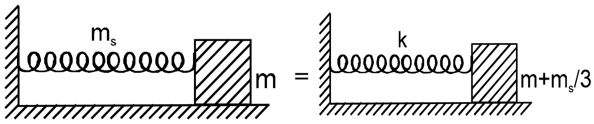

$T = 2\pi\sqrt{\frac{m}{k}}$

**(2)**




$T = 2\pi\sqrt{\frac{m}{k}}$

**(4)** If spring has mass $m_s$ then



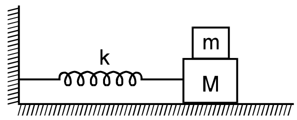

$T = 2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$ **[Not in JEE, for other exams]**

**Ex. 8** A particle of mass 200 g executes a simple harmonic motion. The restoring force is provided by a spring of spring cosntant 80 N/m. Find the time period.

**Sol.** The time period is

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad = 2\pi\sqrt{\frac{200\times10^{-3}\,kg}{80\ N/m}} \qquad = 2\pi \times 0.05\ s = 0.31\ s.$$

**Ex. 9** The friction coefficient between the two blocks shown in figure is μ and the horizontal plane is smooth. (a) If the system is slightly displaced and released, find the time period. (b) Find the magnitude of the frictional force between the blocks when the displacement from the meanposition is x. (c) What can be the maximum amplitude if the upper block does not slip relative to the lower block ?




**Sol.** (a) For small amplitude, the two blocks oscillate together. The angular frequency is

$$\omega = \sqrt{\frac{k}{M+m}}$$

and so the time period $T = 2\pi\sqrt{\frac{M+m}{k}}$ .

(b) The acceleration of the blocks at displacement x from the mean position is

$$a = -\,\omega^2 x = \left(\frac{-kx}{M+m}\right)$$

The resultant force on the upper block is, therefore,

$$ma = \left(\frac{-mkx}{M+m}\right)$$

This force is provided by the friction of the lower block.

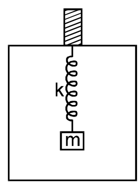
Hence, the magnitude of the frictional force is $\left(\frac{mk\,|x|}{M+m}\right)$

(c) Maximum force of friction required for simple harmonic motion of the upper block is $\frac{mk\,A}{M+m}$ at the extreme positions. But the maximum frictional force can only be μ mg. Hence

$$\frac{mk\,A}{M+m} = \mu\,mg \qquad \text{or,} \qquad A = \frac{\mu(M+m)g}{k}$$

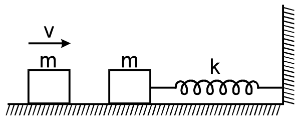
**Ex. 10** A block of mass m is suspended from the ceiling of a stationary elevator through a spring of spring constant k it is in equilibrium. Suddenly, the cable breaks and the elevator starts falling freely. Show that block now executes a simple harmonic motion of amplitude mg/k in the elevator.

**Sol.** When the elevator is stationary, the spring is stretched to support the block. If the extension is x, the tension is kx which should balance the weight of the block.




Thus, x = mg/k. As the cable breaks, the elevator starts falling with acceleration 'g'. We shall work in the frame of reference of the elevator. Then we have to use a psuedo force mg upward on the block. This force will 'balance' the weight. Thus, the block is subjected to a net force kx by the spring when it is at a distance x from the position of unstretched spring. Hence, its motion in the elevator is simple harmonic with its mean position corresponding to the unstretched spring. Initially, the spring is stretched by x = mg/k, where the velocity of the block (with respect to the elevator) is zero. Thus, the amplitude of the resulting simple harmonic motion is mg/k.

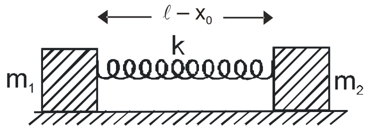
**Ex. 11** The left block in figure collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.



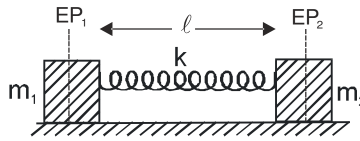

**Sol.** Assuming the collision to last for a small interval only, we can apply the principle of conservation of momentum. The common velocity after the collision is $\frac{v}{2}$. The kinetic energy = $\frac{1}{2}$ (2m) $\left(\frac{v}{2}\right)^2$ = $\frac{1}{4}$ mv$^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is A, the total energy can also be written as $\frac{1}{2}$ kA$^2$. Thus,

$$\frac{1}{2}kA^2 = \frac{1}{4}mv^2, \text{ giving } A = \sqrt{\frac{m}{2k}}\,v.$$

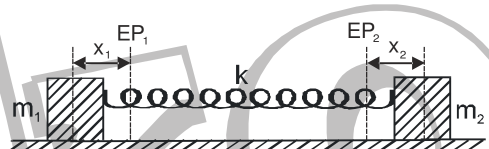
**Ex. 12** Two blocks of mass $m_1$ and $m_2$ are connected with a spring of natural length l and spring constant k. The system is lying on a smooth horizontal surface. Initially spring is compressed by $x_0$ as shown in figure.
Show that the two blocks will perform SHM about their equilibrium position. Also (a) find the time period, (b) find amplitude of each block and (c) length of spring as a function of time.




**Sol.** Here both the blocks will be in equilibrium at the same time when spring is in its natural length. Let $EP_1$ and $EP_2$ be equilibrium positions of block A and B as shown in figure.




Let at any time during oscillations, blocks are at a distance of $x_1$ and $x_2$ from their euilibrium positions.




As no external force is acting on the spring block system

$\therefore \quad (m_1 + m_2)\Delta x_{cm} = m_1x_1 - m_2x_2 = 0 \quad$ or $\quad m_1x_1 = m_2x_2$

For 1st particle, force equation can be written as

$$k(x_1 + x_2) = -m_1\frac{d^2x_1}{dt^2} \quad \text{or,} \quad k\left(x_1 + \frac{m_1}{m_2}x_1\right) = -m_1a_1$$

$$\text{or,} \quad a_1 = -\frac{k(m_1+m_2)}{m_1m_2}x_1 \quad \therefore \quad \omega^2 = \frac{k(m_1+m_2)}{m_1m_2}$$

Hence, $T = 2\pi\sqrt{\frac{m_1m_2}{k(m_1+m_2)}} = 2\pi\sqrt{\frac{\mu}{K}}$ where $\mu = \frac{m_1m_2}{(m_1+m_2)}$ which is known as reduced mass

**Ans (a)**

Similarly time period of 2nd particle can be found. Both will be having the same time period.

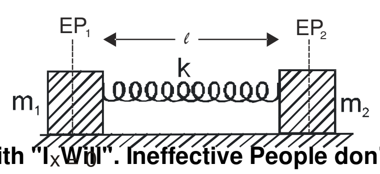
(b) Let the amplitude of blocks be $A_1$ and $A_2$.
$m_1A_1 = m_2A_2$
By energy conservation;

$$\frac{1}{2}k(A_1 + A_2)^2 = \frac{1}{2}k\,x_0^{\,2} \quad \text{or,} \quad A_1 + A_2 = x_0$$

$$\text{or,} \quad A_1 + A_2 = x_0 \quad \text{or,} \quad A_1 + \frac{m_1}{m_2}A_1 = x_0$$

$$\text{or,} \quad A_1 = \frac{m_2x_0}{m_1+m_2} \quad \text{Similarly,} \quad A_2 = \frac{m_1x_0}{m_1+m_2}$$

(c) Let equilibrium position of 1st particle as origin, i.e. x = 0 .
x co-ordinate of particles can be written as



$x_1 = A_1\cos\omega t$ and $x_2 = \ell - A_2\cos\omega t$

Hence, length of spring can be written as;

length $= x_2 - x_1 = \ell - (A_1 + A_2)\cos\omega t$

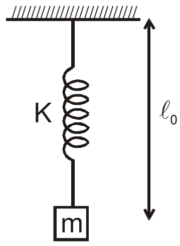
**Q. 2** Block A of mass m is performing SHM of amplitude a. Another block B of mass m is gently placed on A when it passes through mean position and B sticks to A. Find the time period and amplitude of new SHM.

**Ans.** $T = 2\pi\sqrt{\frac{2m}{K}}$ Amplitude $= \frac{a}{\sqrt{2}}$

**Q. 3** Repeat the above problem assuming B is placed on A at a distance $\frac{a}{2}$ from mean position.

**Ans.** $T = 2\pi\sqrt{\frac{2m}{K}}$ , Amplitude $= a\sqrt{\frac{5}{8}}$

**Q. 4** The block is allowed to fall, slowly from the position where spring is in its natural length. Find, maximum extension in the string.

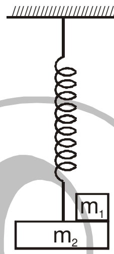



**Ans.** $\frac{mg}{K}$

**Q. 5** In the above problem if block is released from there, what would be maximum extension.

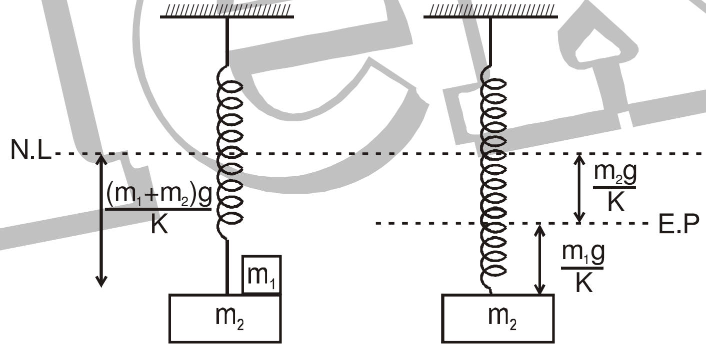
**Ans.** $\frac{2mg}{K}$

**Ex. 13** The system is in equilibrium and at rest. Now mass $m_1$ is removed from $m_2$. Find the time period and amplitude of resultant motion. Spring constant is K.




**Sol.** Initial extension in the spring $x = \frac{(m_1 + m_2)g}{K}$

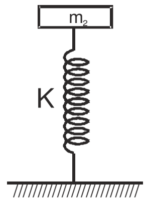
Now, if we remove $m_1$, equilibrium position(E.P.) of $m_2$ will be $\frac{m_2 g}{K}$ below natural length of spring.




At the initial position, since velocity is zero i.e. it is the extreme position.

Hence Amplitude $= \frac{m_1 g}{K}$ Time period $= 2\pi\sqrt{\frac{m_2}{K}}$

**Q. 6** Block of mass $m_2$ is in equilibrium as shown in figure. Another block of mass $m_1$ is kept gently on $m_2$. Find the time period of oscillation and amplitude.




**Ans.** $T = 2\pi\sqrt{\frac{m_1 + m_2}{K}}$ Amplitude $= \frac{m_1 g}{K}$

**Q. 7** Block of mass $m_2$ is in equilibrium and at rest. The mass $m_1$ moving with velocity u



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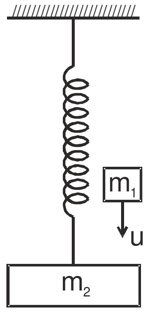
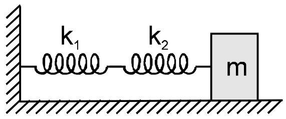
vertically downwards collides with $m_2$ and sticks to it. Find the energy of oscillation.



**Ans.** $\frac{1}{2}\left[\frac{m_1^2u^2}{m_1+m_2}+\frac{m_1^2g^2}{K^2}\right]$

## 8. COMBINATION OF SPRINGS

### 8.1 Series Combination :



Total displacement $x = x_1 + x_2$

Tension in both springs $= k_1x_1 = k_2x_2$

$\therefore$ Equivalent constant in series combination $K_{eq}$ is given by :

$$1/k_{eq} = 1/k_1 + 1/k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

**Note :** **(a)** In series combination, tension is same in all the springs & extension will be different. (If k is same then deformation is also same )

**(b)** In series combination , extension of springs will be reciprocal of its spring constant.

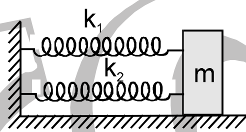
**(c)** Spring constant of spring is reciprocal of its natural length

$\because k \propto 1/\ell$

$\therefore k_1\ell_1 = k_2\ell_2 = k_3\ell_3$

**(d)** If a spring is cut in 'n' pieces then spring constant of one piece will be nk.

### 8.2 Parallel combination :



Extension is same for both springs but force acting will be different.

Force acting on the system = F

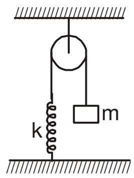
$\therefore \quad F = -(k_1x + k_2x) \quad \Rightarrow \quad F = -(k_1 + k_2)\,x \quad \Rightarrow \quad F = -k_{eq}x$

$\therefore \quad k_{eq} = k_1 + k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$

## 9. Method's to determine time period, angular frequency in S.H.M.

**(a) Force / torque method** **(b) Energy method**

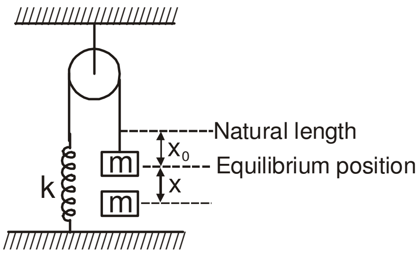
**Ex. 14** The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.



**Sol, (a) Force Method**

Let in equilibrium position of the block, extension in spring is $x_0$.

$\therefore \quad kx_0 = mg \quad$ -- (1)




Now if we displace the block by x in the downward direction, net force on the block towards mean position is

$F = k(x + x_0) - mg \quad = kx \quad$ using (1)

Hence the net force is acting towards mean position and is also proportional to x.So, the particle will perform S.H.M. and its time period would be

$$T = 2\pi\sqrt{\frac{m}{k}}$$

**(b) Energy Method**

Let gravitational potential energy to be zero at the level of the block when spring is in its natural length.

Now at a distance x below that level, let speed of the block be v.

Since total mechanical energy is conserved in S.H.M.

$\therefore \quad -mgx + 1/2kx^2 + 1/2mv^2 = \text{constant}$

Differentiating w.r.t. time, we get $-mgv + kxv + mva = 0$
where a is acceleration.

$\therefore$ $F = ma = -kx + mg$ or $F = -k(x - mg/K)$

This shows that for the motion, force constant is k and equilibrium position is x = mg/K.

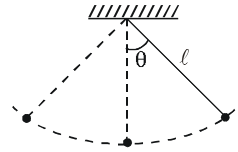
So, the particle will perform S.H.M. and its time period would be $T = 2\pi\sqrt{\dfrac{m}{k}}$

**Q. 8** Solve the above problem if the pulley has a moment of inertia I about its axis and the string does not slip over it.

**Ans.** $2\pi\sqrt{\dfrac{(m + I/r^2)}{k}}$

## 10. SIMPLE PENDULUM

If a heavy point-mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum




Time period of a simple pendulum $T = 2\pi\sqrt{\dfrac{\ell}{g}}$

(some times we can take $g = \pi^2$ for making calculation simple)

**Note :** **(a)** If angular amplitude of simple pendulum is more, then time period

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$$ **(For other exams)**

where $\theta_0$ is in radians.

**(b)** General formula for time period of simple pendulum.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{R} + \frac{1}{\ell}\right)}}$$

**(c)** On increasing length of simple pendulum, time period increases, but time period of simple pendulum of infinite length is 84.6 min which is maximum and is equal to $T = 2\pi\sqrt{\dfrac{R}{g}}$

(Where R is radius of earth)

**(d)** Time period of seconds pendulum is 2 sec and $\ell = 0.993$ m.

**(e)** Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear S.H.M.

**(f)** If time period of clock based on simple pendulum increases then clock will be slow but if time period decrease then clock will be fast.

**(g)** If g remains constant & $\Delta\ell$ is change in length, then $\dfrac{\Delta T}{T}\times 100 = \dfrac{1}{2}\dfrac{\Delta\ell}{\ell}\times 100$

**(h)** If $\ell$ remain constant & $\Delta g$ is change in acceleration then, $\dfrac{\Delta T}{T}\times 100 = -\dfrac{1}{2}\dfrac{\Delta g}{g}\times 100$

**(i)** If $\Delta\ell$ is change in length & $\Delta g$ is change in acceleration due to gravity then,

$$\frac{\Delta T}{T}\times 100 = \left[\frac{1}{2}\frac{\Delta\ell}{\ell} - \frac{1}{2}\frac{\Delta g}{g}\right]\times 100$$

**Ex. 15** A simple pendulum of length 40 cm oscillates with an angular amplitude of 0.04 rad. Find (a) the time period, (b) the linear amplitude of the bob, (c) the speed of the bob when the string makes 0.02 rad with the vertical and (d) the angular acceleration when the bob is in momentary rest. Take g = 10 m/s².

**Sol.** (a) The angular frequency is $\omega = \sqrt{g/\ell} = \sqrt{\frac{10\,m/s^2}{0.4m}} = 5\ s^{-1}$

the time period is $\frac{2\pi}{\omega} = \frac{2\pi}{5s^{-1}} = 1.26$ s.

(b) Linear amplitude = 40 cm × 0.04 = 1.6 cm

(c) Angular speed at displacement 0.02 rad is

$\Omega = (5\ s^{-1})\sqrt{(0.04)^2 - (0.02)^2}$ rad = 0.17 rad/s.

where speed of the bob at this instant

= (40 cm) × $0.175^{-1}$ = 6.8 cm/s.

(d) At momentary rest, the bob is in extreme position.

Thus, the angular acceleration

$\alpha = (0.04\ rad)\ (25\ s^{-2}) = 1\ rad/s^2$.

**10.1 Time Period of Simple Pendulum in accelerating Reference Frame :**

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}} \text{ where}$$

$g_{eff.}$ = Effective acceleration due to gravity in reference system = $|\vec{g} - \vec{a}|$

$\vec{a}$ = acceleration of the point of suspension w.r.t. ground.

**Condition for applying this formula:** $|\vec{g} - \vec{a}|$ = constant

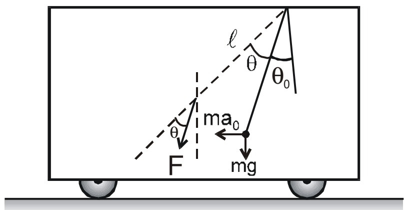
**Ex. 16** A simple pendulum is suspended from the ceiling of a car accelerating uniformly on a horizontal road. If the acceleration is $a_0$ and the length of the pendulum is $\ell$, find the time period of small oscillations about the mean position.

**Sol.** We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a psuedo force $ma_0$ on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If θ be the angle made by the string with the vertical, the tension, weight and the psuedo force will add to zero in this position.

Hence, resultant of mg and $ma_0$ (say $F = m\sqrt{g^2 + a_0^2}$ ) has to be along the string.

$\therefore$ $\tan\theta_0 = \frac{ma_0}{mg} = \frac{a_0}{g}$




Now, suppose the string is further deflected by an angle θ as shown in figure.

Now, restoring torque can be given by

$(F \sin\theta)\ \ell = -m\ell^2\alpha$

Substituting F and using $\sin\theta = \theta$ , for small θ.

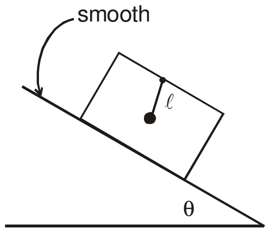
$(m\sqrt{g^2 + a_0^2})\ \ell\theta = -m\ell^2\alpha$

or, $\alpha = -\frac{\sqrt{g^2 + a_0^2}}{\ell}\theta$ so; $\omega^2 = \frac{\sqrt{g^2 + a_0^2}}{\ell}$

This is an equation of simple harmonic motion with time period

$$T = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{\ell}}{(g^2 + a_0^2)^{1/4}}$$

**Q. 9** A block is placed on a smooth inclined plane and it is free to move.
A simple pendulum is attached in the block. Find its time period.




**Ans.** $T = 2\pi\sqrt{\dfrac{\ell}{g\cos\theta}}$

**10.2 If forces other then $m\vec{g}$ acts then:**

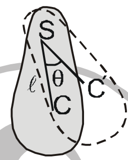
$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}} \quad \text{where} \quad g_{eff.} = \left|\vec{g} + \frac{\vec{F}}{m}\right|$$

$\vec{F}$ = constant force acting on 'm'.

**Ex. 17** A simple pendulum of length '$\ell$' and having bob of mass 'm' is doing angular SHM inside water. A constant buoyant force equal to half the weight of the bob is acting on the ball. Find the time period of oscillations?

**Sol.** Here $g_{eff.} = g - \dfrac{mg/2}{m} = g/2$. Hence $T = 2\pi\sqrt{\dfrac{2\ell}{g}}$

## 11. COMPOUND PENDULUM / PHYSICAL PENDULUM




When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.
C = Initial position of center of mass
C′ = Position of center of mass after time t
S = Point of suspension
$\ell$ = Distance between point of suspension and center of mass (it remains constant during motion)
For small angular displacement "θ" from mean position
The restoring torque is given by

$\tau = -mgl\theta$

or, $I\alpha = -mgl\theta$ where, I = Moment of inertia about point of suspension.

or, $\alpha = -\dfrac{mgl}{I}\theta$ or, $\omega^2 = \dfrac{mgl}{I}$

Time period, $T = 2\pi\sqrt{\dfrac{I}{mg\ell}}$ $I = I_{CM} + m\ell^2$

where $I_{CM}$ = moment of inertia relative to the axis which passes from the center of mass & parallel to the axis of oscillation.

$$T = 2\pi\sqrt{\frac{I_{CM} + m\ell^2}{mg\ell}} \quad \text{where} \quad I_{CM} = mk^2$$

k = gyration radius (about axis passing from centre of mass)

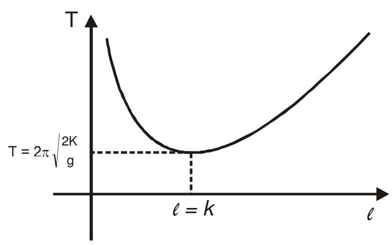
$$T = 2\pi\sqrt{\frac{mk^2 + m\ell^2}{mg\ell}} \qquad T = 2\pi\sqrt{\frac{k^2 + \ell^2}{\ell g}} = 2\pi\sqrt{\frac{L_{eq}}{g}}$$

$$L_{eq} = \frac{k^2}{\ell} + \ell$$ = equivalent length of simple pendulum ;

**T is minimum when $\ell$ = k.**

$T_{min} = 2\pi\sqrt{\dfrac{2K}{g}}$

**Graph of T vs $\ell$**



**Ex. 18** A uniform rod of length 1.00 m is suspended through an end and is set into oscillation with small amplitude under gravity. Find the time period of oscillation. ($g = 10\ m/s^2$)

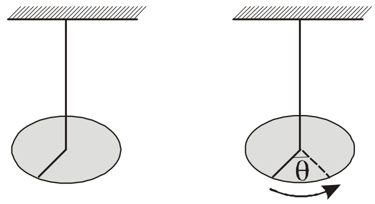
**Sol.** For small amplitude the angular motion is nearly simple harmonic and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mg(\ell/2)}} \qquad = 2\pi\sqrt{\frac{(m\ell^2/3)}{mg(\ell/2)}}$$

$$= 2\pi\sqrt{\frac{2\ell}{3g}} \qquad = 2\pi\sqrt{\frac{2\times 1.00\,m}{3\times 10\,m/s^2}} \qquad = \frac{2\pi}{\sqrt{15}}\ s.$$

## 12. TORSIONAL PENDULUM

In torsional pendulum, an extended object is suspended at the centre by a light torsion wire. A torsion wire is essentially inextensible, but is free to twist about its axis. When the lower end of the wire is rotated by a slight amount, the wire applies a restoring torque causing the body to oscillate rotationally when released.




The restoring torque produced is given by

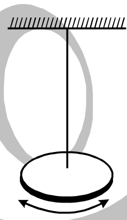
$\tau = -C\theta$ where, C = Torsional constant

or, $I\alpha = -C\theta$ where, I = Moment of inertia about the vertical axis.

or, $\alpha = -\frac{C}{I}\theta$ $\therefore$ Time Period, $T = 2\pi\sqrt{\frac{I}{C}}$

**Ex. 19** A uniform disc of radius 5.0 cm and mass 200 g is fixed at its centre to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

**Sol.** The situation is shown in figure. The moment of inertia of the disc about the wire is



$$I = \frac{mr^2}{2} = \frac{(0.200\,kg)(5.0\times 10^{-2}m)^2}{2} \qquad = 2.5\times 10^{-4}\ kg - m^2.$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad \text{or,} \qquad C = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4\pi^2(2.5\times 10^{-4}kg - m^2)}{(0.20\,s)^2} \qquad = 0.25\ \frac{kg - m^2}{s^2}.$$

## 13. SUPERPOSITION OF TWO SHM'S.

### 13.1 In same direction and of same frequency.

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin(\omega t + \theta)$ , then resultant displacement

$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin(\omega t + \theta) = A \sin(\omega t + \phi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$ & $\phi = \tan^{-1}\left[\frac{A_2\sin\theta}{A_1 + A_2\cos\theta}\right]$

If $\theta = 0$, both SHM's are in phase and $A = A_1 + A_2$

If $\theta = \pi$, both SHM's are out of phase and $A = |A_1 - A_2|$

The resultant amplitude due to superposition of two or more than two SHM's of this case can also be found by phasor diagram also.

### 13.2 In same direction but are of different frequencies.

$x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$

This resultant motion is not SHM.

### 13.3 In two perpendicular directions.

$x = A \sin \omega t$

$y = B \sin(\omega t + \theta)$

**Case (i) :** If $\theta = 0$ or $\pi$ then $y = \pm$ (B/A) x. So path will be straight line & resultant displacement will be $r = (x^2 + y^2)^{1/2} = (A^2 + B^2)^{1/2} \sin\omega t$

which is equation of SHM having amplitude $\sqrt{A^2 + B^2}$

**Case (ii) :** If $\theta = \frac{\pi}{2}$ then. $x = A \sin \omega t$

$y = B \sin (\omega t + \pi/2) = B \cos \omega t$

so, resultant will be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$. i.e. equation of an ellipse and if A = B, then superposition will be an equation of circle.

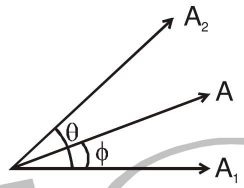
**13.4 Superposition of SHM's along the same direction (using phasor diagram)**

If two or more SHM's are along the same line, their resultant can be obtained by vector addition by making phasor diagram.

**1.** Amplitude of SHM is taken as length(magnitude) of vector.

**2.** Phase difference between the vectors is taken as the angle between these vectors. The magnitude of resultant of vector's give resultant amplitude of SHM and angle of resultant vector gives phase constant of resultant SHM.

**For example;**




$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin (\omega t + \theta)$

If equation of resultant SHM is taken as $x = A \sin (\omega t + \phi)$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$$

$$\tan \phi = \frac{A_2 \sin\theta}{A_1 + A_2 \cos\theta}$$

**Ex. 20** Find the amplitude of the simple harmonic motion obtained by combining the motions

$x_1 = (2.0 \text{ cm}) \sin\omega t$

and $x_2 = (2.0 \text{ cm}) \sin (\omega t + \pi/3)$.

**Sol.** The two equations given represent simple harmonic motions along X-axis with amplitudes $A_1 = 2.0$ cm and $A_2 = 2.0$ cm. The phase differnce between the two simple harmonic motions is $\pi/3$. The resultant simple harmonic motion will have an amplitude A given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta} = \sqrt{(2.0\text{cm})^2 + (2.0\text{ cm})^2 + 2(2.0\text{cm})^2\cos\frac{\pi}{3}}$$

$= 3.5$ cm

**Ex. 21** $x_1 = 3 \sin \omega t$

$x_2 = 4 \cos \omega t$

Find (i) amplitude of resultant SHM. (ii) equation of the resultant SHM.

**Sol.** First write all SHM's in terms of sine functions with positive amplitude. Keep "ωt" with positive sign.

$\therefore$ $x_1 = 3 \sin \omega t$

$x_2 = 4 \sin (\omega t + \pi/2)$

$$A = \sqrt{3^2 + 4^2 + 2\times3\times4\cos\frac{\pi}{2}} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\tan \phi = \frac{4\sin\frac{\pi}{2}}{3 + 4\cos\frac{\pi}{2}} = \frac{4}{3} \qquad \phi = 53^\circ$$

equation $x = 5 \sin (\omega t + 53^\circ)$



**Ex. 22** $x_1 = 5 \sin(\omega t + 30^\circ)$

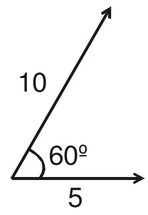
$x_2 = 10 \cos(\omega t)$

Find amplitude of resultant SHM.

**Sol.** $x_1 = 5 \sin(\omega t + 30^\circ)$

$x_2 = 10 \sin(\omega t + \frac{\pi}{2})$

Phasor diagram




$$A = \sqrt{5^2 + 10^2 + 2 \times 5 \times 10 \cos 60^\circ}$$

$$= \sqrt{25 + 100 + 50} \qquad = \sqrt{175} \qquad = 5\sqrt{7}$$

**Q. 10** $x_1 = 5 \sin \omega t$

$x_2 = 5 \sin(\omega t + 53^\circ)$

$x_3 = -10 \cos \omega t$

Find amplitude of resultant SHM

**Ans.** 10

**Ex. 23** A particle is subjected to two simple harmonic motions

$x_1 = A_1 \sin \omega t$

and $x_2 = A_2 \sin(\omega t + \pi/3)$.

Find (a) the displacement at t = 0, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle.

**Sol.** (a) At $t = 0$, $x_1 = A_1 \sin \omega t = 0$

and $x_2 = A_2 \sin(\omega t + \pi/3)$

$$= A_2 \sin(\pi/3) = \frac{A_2\sqrt{3}}{2}.$$

Thus, the resultant displacement at t = 0 is

$$x = x_1 + x_2 = A_2 \frac{\sqrt{3}}{2}$$

(b) The resultant of the two motions is a simple harmonic motion of the same angular frequency $\omega$. The amplitude of the resultant motion is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\pi/3)} \qquad = \sqrt{A_1^2 + A_2^2 + A_1A_2}.$$

The maximum speed is

$$u_{max} = A\,\omega = \omega\sqrt{A_1^2 + A_2^2 + A_1A_2}$$

(c) The maximum acceleration is

$$a_{max} = A\,\omega^2 = \omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}.$$

**Ex. 24** A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.

**Sol.** Let the amplitudes of the individual motions be A each. The resultant amplitude is also A. If the phase difference between the two motions is $\delta$,

$$A = \sqrt{A^2 + A^2 + 2A.\,A.\cos\delta} \qquad \text{or,} \qquad = A\sqrt{2(1+\cos\delta)} = 2A\cos\frac{\delta}{2}$$

or, $\cos\frac{\delta}{2} = \frac{1}{2}$ or, $\delta = 2\pi/3$.

