1. COMPLEX NUMBERS

1. DEFINITION: Complex numbers are defined as expressions of the form \( a + \text{i}b \) where \( a, b \in \mathbb{R} \) and \( \text{i} = \sqrt{-1}. \) It is denoted by \( z \) i.e. \( z = a + \text{i}b \). ’a’ is called as real part of \( z \) (Re \( z \)) and ’b’ is called as imaginary part of \( z \) (Im \( z \)).

EVERY COMPLEX NUMBER CAN BE REGARDED AS

<table>
<thead>
<tr>
<th>Purely real</th>
<th>Purely imaginary</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>if ( b = 0 )</td>
<td>if ( a = 0 )</td>
<td>if ( b \neq 0 )</td>
</tr>
</tbody>
</table>

Note:
(a) The set \( \mathbb{R} \) of real numbers is a proper subset of the Complex Numbers. Hence the Complete Number system is \( \mathbb{C} \subset \mathbb{R} \subset \mathbb{C} \).

(b) Zero is both purely real as well as purely imaginary but not imaginary.

(c) \( \sqrt{a} \sqrt{b} = \sqrt{ab} \) only if at least one of either a or b is non-negative. www.MathsBySuhag.com

2. CONJUGATE COMPLEX:
If \( z = a + \text{i}b \) then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by \( \bar{z} \). i.e. \( \bar{z} = a - \text{i}b \).


(i) \( z + \bar{z} = 2 \text{Re}(z) \) (ii) \( z - \bar{z} = 2\text{Im}(z) \) (iii) \( z \bar{z} = a^2 + b^2 \) which is real

(iv) If \( z \) lies in the 1st quadrant then \( \bar{z} \) lies in the 4th quadrant and \( -z \) lies in the 2nd quadrant.

3. ALGEBRAIC OPERATIONS:
The algebraic operations on complex numbers are similar to those on real numbers treating \( \text{i}^2 = -1 \).

4. EQUALITY IN COMPLEX NUMBER:
Two complex numbers \( z_1 = a_1 + \text{i}b_1 \) & \( z_2 = a_2 + \text{i}b_2 \) are equal if and only if their real & imaginary parts coincide.

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS:
(a) Cartesian Form (Geometric Representation):
Every complex number \( z = x + \text{i}y \) can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair \((x, y)\). Length \( OP \) is called modulus of the complex number denoted by \( |z| \) & \( \theta \) is called the argument or amplitude

e.g. \( |z| = \sqrt{x^2 + y^2} \)

\( \theta = \tan^{-1} \frac{y}{x} \) (angle made by OP with positive x-axis)

NOTE: (i) \( |z| \) is always non negative. Unlike real numbers \( |z| = \sqrt{z \overline{z}} \) is not correct

(ii) Argument of a complex number is a many valued function. If \( \theta \) is the argument of a complex number then \( 2 \pi n + \theta \) : \( n \in \mathbb{N} \) will also be the argument of that complex number. Any two arguments of a complex number differ by \( 2\pi n \).

(iii) The unique value of \( \theta \) such that \( -\pi < \theta \leq \pi \) is called the principal value of the argument.

(iv) Unless otherwise stated, \( \text{amp} z \) implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number \( 0 + 0 \text{i} \) the argument is not defined and this is the only complex number which is given by its modulus.

6. IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE:

(a) \( z + \bar{z} = 2 \text{Re}(z) \)

(b) \( z - \bar{z} = 2\text{Im}(z) \)

(c) \( z \bar{z} = a^2 + b^2 \) which is real

(d) \( z \overline{z} = |z|^2 \)

(e) \( \text{amp}(z \overline{z}) = \text{amp} z + \text{amp} \bar{z} \)

(f) \( |z| = |\overline{z}| \)

7. TRIGONOMETRIC REPRESENTATION OF A COMPLEX:
Every complex number can be considered as if it is the position vector of that point. If the point \( P \) represents the complex number \( z \) then, \( \overrightarrow{OP} = z \) & \( |\overrightarrow{OP}| = |z| \).

NOTE: (i) If \( \text{opp} z = r e^{\text{i}\theta} \) then \( \text{opp} \overline{z} = r e^{-\text{i}\theta} \) & \( z \bar{z} = r^2 \).

(ii) If \( A, B, C \) and \( D \) are four points representing the complex numbers \( z_1, z_2, z_3 \) & \( z_4 \) then

\( AB \parallel CD \) if \( \frac{z_4 - z_3}{z_2 - z_1} \) is purely real;

\( AB \perp CD \) if \( \frac{z_4 - z_3}{z_2 - z_1} \) is purely imaginary

(iii) If \( z_1, z_2, z_3 \) are the vertices of an equilateral triangle where \( z_0 \) is its circumcentre then

\( a) \ z_0 = \frac{z_1 + z_2 + z_3}{3} \) & \( z_1 = z_2 = z_3 \)

\( b) \ z_0 = \frac{z_1 + z_2^* + z_3^*}{3} \) & \( z_1 = z_2 = z_3 \)

8. DEMOIVRE’S THEOREM:
Statement: \( \cos n \theta + i \sin n \theta \) is the value or one of the values of \( (\cos \theta + i \sin \theta)^n \) \( \forall n \in \mathbb{Q} \).

The theorem is very useful in determining the roots of any complex quantity.

Note: Continued product of the roots of a complex quantity should be determined using theory of equations.

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9. CUBE ROOT OF UNITY (i) The cube roots of unity are 1, \(\frac{-1+\sqrt{3}i}{2}\), \(\frac{-1-\sqrt{3}i}{2}\).

(ii) If \(w\) is one of the imaginary cube roots of unity then \(1+w^2+w^3=0\). In general \(1+w^n+w^2=0\) where \(r\) is real and \(n\) is not the multiple of 3.

(iii) In polar form the cube roots of unity are:
\[
\cos \frac{2\pi}{3} \pm i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} \pm i \sin \frac{4\pi}{3}
\]

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.


(v) The following factorisation should be remembered:

\[
(a \cos \theta + b \sin \theta)^n = \cos n\theta + i \sin n\theta
\]


If \(1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}\) are the \(n\) \(n\)th root of unity then:

(i) They are in G.P. with common ratio \(\cos \frac{2\pi}{n}\) & \(\sin \frac{2\pi}{n}\)

(ii) \(1 + \alpha^n + \alpha^{2n} + \ldots + \alpha^{(n-1)n} = 0\) if \(p\) is not an integral multiple of \(n\)

(iii) \((1 - \alpha^n)(1 - \alpha^{2n}) \ldots (1 - \alpha^{(n-1)n}) = n\) & \(n\) is a perfect square,

(iv) \((1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-1}) = 0\) if \(n\) is even and 1 if \(n\) is odd.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED:

(i) \(\cos \theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin \left(\frac{n+1}{2}\right)\theta}{\sin \left(\frac{\theta}{2}\right)}\).

(ii) \(\sin \theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin \left(\frac{n+1}{2}\right)\theta}{\sin \left(\frac{\theta}{2}\right)}\).

Note: If \(\theta \equiv 2\pi n\) then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS:

(A) If \(z_1, z_2, z_3\) are two complex numbers then the complex number \(z = \frac{az_1 + bz_2 + cz_3}{m+n} \) divides the joins of \(z_1\) & \(z_2\) in the ratio \(m:n\).

Note(i): If \(a, b, c\) are three real numbers such that \(az_1 + bz_2 + cz_3 = 0\), then the complex numbers \(z_1, z_2, z_3\) are collinear.

(ii) If the vertices \(A, B, C\) of a \(\Delta\) represent the complex nos. \(z_1, z_2, z_3\) respectively, then:

(a) Centroid of the \(\Delta ABC = z_1 + z_2 + z_3\).

(b) Orthocentre of the \(\Delta ABC = \frac{(a \sec \theta) z_1 + (b \sec \theta) z_2 + (c \sec \theta) z_3}{a \sec \theta + b \sec \theta + c \sec \theta} \) \(\tan A + z_2 \tan B + z_3 \tan C\).

(c) Incentre of the \(\Delta ABC = \frac{(az_1 + bz_2 + cz_3)}{a+b+c} \) + \(a + b + c\).

(d) Circumcentre of the \(\Delta ABC = (I)\sin 2A + Z_1, \sin 2B + Z_2, \sin 2C \) \(\cos 2A + \sin 2B + \sin 2C\).

(B) amp(z) = \(\theta\) is a ray emanating from the origin inclined at an angle \(\theta\) to the \(x\)-axis.

(C) \(|z - a| = |z - b|\) is the perpendicular bisector of the line joining \(a\) & \(b\).

(D) The equation of a line joining \(z_1\) & \(z_2\) is given by \(z = z_1 + t(z_2 - z_1)\) where \(t\) is a parameter.

(E) \(z = z_1(1 + it)\) where \(t\) is a real parameter is a line through the point \(z_1\) perpendicular to \(oz_1\).

(F) The equation of a line passing through \(z_1\) & \(z_2\) can be expressed in the determinant form as

\[
\begin{vmatrix}
z & z_1 & z_2 \\
z_1 & z_1 & z_2 \\
z_2 & z_1 & z_2 \\
\end{vmatrix} = 0.
\]

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5. Remember that a quadratic equation cannot have three different roots & if it has, it becomes an identity.

6. Consider the quadratic expression, \( y = ax^2 + bx + c \), \( a \neq 0 \) & \( a, b, c \in \mathbb{R} \) then

(i) The graph between \( x, y \) is always a parabola. If \( a > 0 \) then the shape of the parabola is concave upwards & if \( a < 0 \) then the shape of the parabola is concave downwards.

(ii) \( \forall x \in \mathbb{R}, y > 0 \) only if \( a > 0 \) & \( b^2 - 4ac < 0 \) (figure 3).

(iii) \( \forall x \in \mathbb{R}, y < 0 \) only if \( a < 0 \) & \( b^2 - 4ac < 0 \) (figure 6).

Carefully go through the 6 different shapes of the parabola given below.

\[ \begin{align*}
\text{(i)} & \quad y = ax^2 + bx + c, \quad a \neq 0 \quad & \text{Roots are real} & \text{&} & \text{&} \\
\text{(ii)} & \quad y = ax^2 + bx + c, \quad a > 0 \quad & \text{Roots are real} & \text{&} & \text{&} \\
\text{(iii)} & \quad y = ax^2 + bx + c, \quad a < 0 \quad & \text{Roots are complex} & \text{&} & \text{&}
\end{align*} \]

7. SOLUTION OF QUADRATIC INEQUALITIES:

\[ ax^2 + bx + c > 0 \quad (a \neq 0). \]

(i) If \( D > 0 \), then the equation \( ax^2 + bx + c = 0 \) has two different roots \( x_1 < x_2 \).

Then \( a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_2, \infty) \)

(ii) If \( D = 0 \), then roots are equal, i.e., \( x_1 = x_2 \).

In that case \( a > 0 \Rightarrow x \in \Phi \)

(iii) Inequalities of the form \( P(x) > 0 \) can be quickly solved using the method of intervals.

8. MAXIMUM & MINIMUM VALUE of \( y = ax^2 + bx + c \) occurs at \( x = -(b/2a) \) according as \( a > 0 \) or \( a < 0 \).

\( y = \frac{4ac - b^2}{4a} \) if \( a > 0 \) & \( y = \frac{4ac - b^2}{4a} \) if \( a < 0 \).

9. COMMON ROOTS OF 2 QUADRATIC EQUATIONS [ONLY ONE COMMON ROOT] :

Let \( \alpha \) be the common root of \( ax^2 + bx + c = 0 \) & \( a'x^2 + b'x + c' = 0 \). Therefore \( \alpha^2 + bx + c = 0 \) & \( a'\alpha^2 + b'\alpha + c' = 0 \).

By Cramer's Rule \( \frac{\alpha^2 + bx + c}{a'\alpha^2 + b'\alpha + c'} = \frac{1}{ab' - a'b} \) Therefore, \( \alpha = \frac{ca' - c'a^2 - bc'c}{ab' - a'b} \). So the condition for a common root is \( (ca' - c'a^2 = (ab' - a'b)bc' - bc'c) \).

10. The condition that a quadratic function \( f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \) may be resolved into two linear factors is that:

\[ abc + 2gh - a^2 - b^2 - c^2 = 0 \text{ or } \frac{h}{b} \frac{g}{c} \frac{f}{a} = 0. \]

11. THEORY OF EQUATIONS:

If \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \) are the roots of the equation;

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0 \text{ where } a_n, a_{n-1}, \ldots, a_0 \text{ are all real & } a_n \neq 0 \text{ then, } \sum \alpha_i = -\frac{a_1}{a_n}, \]

\[ \sum \alpha_i \alpha_j = \frac{a_{n-1}}{a_n}, \quad \ldots, \quad \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^{n} \frac{a_0}{a_n}. \]

Note:

(i) If \( a < 0 \) is a root of the equation \( f(x) = 0 \), then the polynomial \( x \) is exactly divisible by \( x + a \) or \( x - a \) is a factor of \( f(x) \) & conversely.

(ii) Every equation of nth degree \( (n \geq 1) \) has exactly \( n \) roots & if the equation has more than \( n \) roots, it is an identity.

(iii) If the coefficients of the equation \( f(x) = 0 \) are all real & \( \alpha + i\beta \) is its root, then \( \alpha - i\beta \) is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) If the coefficients in the equation are all rational & \( \sqrt{\alpha} + \sqrt{\beta} \) is one of its roots, then \( \sqrt{\alpha} - \sqrt{\beta} \) is also a root where \( \alpha, \beta \in \mathbb{Q} \) & \( \beta \) is not a perfect square.

(v) If there be any two real numbers \( \alpha \) & \( \beta \) such that \( f(\alpha) \) & \( f(\beta) \) are of opposite signs, then \( f(x) = 0 \) must have at least one real root between \( \alpha \) and \( \beta \).

(vi) Every equation \( f(x) = 0 \) of degree odd has at least one real root of a sign opposite to that of its last term.

12. LOCATION OF ROOTS:

\[ \text{Let } f(x) = ax^2 + bx + c, \quad a > 0 \text{ & } a, b, c \in \mathbb{R}. \]

(i) Conditions for both the roots of \( f(x) = 0 \) to be greater than a specified number \( \alpha \) are \( b^2 - 4ac > 0; \ f(\alpha) > 0 \text{ & } (\alpha - b2a) > d. \)

(ii) Conditions for both roots of \( f(x) = 0 \) to lie on either side of the number \( \alpha \) (in other words the number \( \alpha \) lies between the roots of \( f(x) = 0 \)) is \( f(d) < 0 \).

(iii) Conditions for exactly one root of \( f(x) = 0 \) to lie in the interval \((c, d)\) i.e. \( d < x < c \) are \( b^2 - 4ac < 0 \text{ & } f'(d), f(c) < 0. \)

(iv) Conditions that both roots of \( f(x) = 0 \) to be confined between the numbers \( p \) & \( q \) are \( (p < q) \; b'2 - 4ac > 0; \; f(p) > 0 \text{ & } f(q) > 0 \text{ & } p < c < q < b'2a < q. \)

13. LOGARITHMIC INEQUALITIES

(i) For \( a > 1 \) the inequality \( 0 < x < y \) & \( \log_a x < \log_a y \) are equivalent.

(ii) For \( 0 < a < 1 \) the inequality \( 0 < x < y \) \& \( \log_a x > \log_a y \) are equivalent.

(iii) If \( a > 1 \) then \( \log_a x > p \Rightarrow 0 < x < a^p \)

(iv) If \( a > 1 \) then \( \log_a x > p \Rightarrow x > a^p \)

(v) If \( 0 < a < 1 \) then \( \log_a x < p \Rightarrow x > a^p \)

(vi) If \( 0 < a < 1 \) then \( \log_a x < p \Rightarrow 0 < x < a^p \)


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**3. Sequence & Progression (AP, GP, HP, AGP, Spl. Series)**

**Definition:** A sequence is a set of terms in a definite order with a rule for obtaining the terms.

e.g. 1, 1/2, 1/3, …., 1/n, …… is a sequence.

**AN ARITHMETIC PROGRESSION (AP):** AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If it is the first term & the common difference, then AP can be written as \( a, a + d, a + 2d, \ldots, a + (n - 1) d, \ldots \), \( n^{th} \) term of this AP is \( a + (n - 1)d \).

\( n \text{ th} \) term of this AP is \( a + (n - 1)d \), where \( d = a_n - a_1 \). The sum of the first \( n \) terms of the AP is given by \( S_n = \frac{n}{2} [2a + (n - 1)d] = \frac{n}{2} [a + l] \), where \( l \) is the last term.

**Notes:**

(i) If each term of an AP is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in an AP can be taken as \( a - d, a, a + d \); four numbers in an AP can be taken as \( a - 3d, a - d, a + d, a + 3d \); five numbers in an AP are \( a - 2d, a - d, a + d, a + 2d \) & six terms in an AP are \( a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d \) etc.

(iii) The common difference can be zero; positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last term.
GEOMETRIC PROGRESSION (GP):

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceding terms multiplied by a constant. Thus in a GP, the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the series & is obtained by dividing any term by that which immediately precedes it. Therefore, a, ar, ar², ar³, .... is a GP with a as the first term & r as common ratio.

\[ S_n = a \frac{r^n - 1}{r - 1} \quad \text{if } r \neq 1 \]

GEOMETRIC MEANS:

If a, b, c are in GP, then b is the GM between a & c. The product of n GMs between a & b is equal to the n-th power of the single GM between a & b.

\[ \prod_{i=1}^{n} G_i = (G)^n \] where G is the single GM between a & b.

HARMONIC MEAN:

If a, b, c are in HP, b is the HM between a & c, then \[ b = \frac{2ac}{a + c} \]

THEOREM:

If A, G, H are respectively AM, GM, HM between a & b both being unequal & positive then,

\[ i) \quad G^2 = AH \]
\[ ii) \quad A > G > H \quad \text{if} G > 0 \]

Note that A, G, H constitute a GP.

ARITHMETIC-GEOMETRIC SERIES:

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the ARITHMETICO-GEOMETRIC SERIES. E.g. \( 1 + 3x + 3x^2 + x^3 + ... \) Here 1, 3, 5, ... are in AP & 1, x, x², x³, ... are in GP.

SIGMA NOTATIONS:

\[ \sum_{r=1}^{n} ar = \frac{a}{1-r} (1-r^n) \quad \text{if } |r| < 1 \]

\[ \lim_{n \to \infty} r^n = 0 \quad \text{if } |r| < 1 \]

RESULTS:

\[ \sum_{r=1}^{n} r = \frac{n(n+1)}{2} \] (sum of the first n natural nos.)

\[ \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \] (sum of the squares of the first n natural numbers)

\[ \sum_{r=1}^{n} r^3 = \left( \frac{n(n+1)}{2} \right)^2 \] (sum of the cubes of the first n natural numbers)

\[ \sum_{r=1}^{n} r^4 = \frac{n^4(n+1)^4}{30} \] (sum of the fourth powers of the first n natural numbers)

METHOD OF DIFFERENCE:

If \( T_1, T_2, T_3, ..., T_n \) are the terms of a sequence, then sometimes the terms \( T_{n-1}, T_{n-2}, T_{n-3}, ... \) constitute an AP/GP. \( n^{th} \) term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Remember that to find the sum of n terms of a series each term of which is composed of r factors in AP, the first factors of several terms being in the same AP, we write down the nth term, affix the next factor & carry on. Divide by the number of factors thus increased & by the common difference & add a constant. Determine the value of the constant by applying the initial conditions.

4. PERMUTATION AND COMBINATION

DEFINITIONS:

1. PERMUTATION: Each of the arrangements in a definite order which can be made by taking some or all of a number of things is called a PERMUTATION.

2. COMBINATION: Each of the groups or selections which can be made by taking some or all of a number of things without reference to the order of the things in each group is called a COMBINATION.

FUNDAMENTAL PRINCIPLE OF COUNTING:

If an event can occur in ‘m’ different ways, following which another event can occur in ‘n’ different ways, then the total number of different ways of simultaneous occurrence of both events in a definite order is \( m \times n \).

RESULTS:

\( P_r \) denotes the number of permutations of n different things, taking r at a time, then

\[ P_r = n(n-1)(n-2)......(n-r+1) = \frac{n!}{(n-r)!} \] where n! denotes the product of n positive integers starting from 1.
The number of ways in which \( m + n + p = N \) things, where \( p \) are alike can be resolved as a product of two
\[ \frac{n!}{(n-1)!} \] factors is
\[ \frac{1}{2!(a+1)(b+1)(c+1)...} \] if \( N \) is not a perfect square
\[ \frac{1}{a+1}(b+1)(c+1)...+1 \] if \( N \) is a perfect square

**DEARRANGEMENT:** Number of ways in which \( n \) letters can be placed in \( n \) directed letters so that no letter goes into its own envelope is
\[ n! \left( \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ...... +(-1)^{n-1} \right) \]

**5. DETERMINANT**

1. The symbol \( \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} \) is called the determinant of order two.

   Its value is given by:
   \[ D = a_{11}b_{22}c_{33} - a_{12}b_{23}c_{31} + a_{13}b_{21}c_{32} \]

2. The symbol \( \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} \) is called the determinant of order three.

   Its value can be found as:
   \[ D = a_{11}b_{22}c_{33} - a_{12}b_{23}c_{31} + a_{13}b_{21}c_{32} \]

or
   \[ D = a_{11}b_{22}c_{33} - b_{11}a_{22}c_{33} + c_{11}a_{22}b_{33} ...... \]

3. Following examples of short hand writing large expressions are:
   (i) The lines:
   \[ a_i x + b_i y + c_i = 0, \ldots, 2 \]
   \[ a_i x + b_i y + c_i = 0, \ldots, 2 \]
   \[ a_i x + b_i y + c_i = 0, \ldots, 2 \]
   \[ a_i x + b_i y + c_i = 0, \ldots, 2 \]

   (ii) If the conditions of the consistency of three simultaneous linear equations in 2 variables.
   \[ ax^2 + 2bxy + by^2 + 2gx + 2fy + c = 0 \]
   \[ abc + 2ghi - a^2b - b^2c - c^2a = 0 \]

   (iii) Condition of the area of a triangle whose vertices are \( (x_i, y_i); \) \( r = 1, 2, 3 \) is:
   \[ \frac{1}{2} | x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 | \]
   If \( D = 0 \) then the three points are collinear.
Let \( D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \) and the minor of \( b \) is \( \begin{vmatrix} d & f \\ g & i \end{vmatrix} \). Hence a determinant of order two will have "4 minors" for a determinant of order three will have "9 minors".

5. **Cofactor** : If \( M \) represents the minor of some typical element then the cofactor is defined as : 

\[ C = (-1)^{i+j} M \]

where \( i \) & \( j \) denotes the row & column in which the particular element lies. Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as : 

\[ D = a_1M_1 - a_2M_2 + a_3M_3 \]

6. **Properties of Determinants**

- **P-1** : The value of a determinant remains unaltered if the rows & columns are inter changed. e.g. 

\[ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \]

if \( D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \), then \( D' = \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \) are transpose of each other. If \( D' = -D \) then it is a **skew symmetric** determinant but \( D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \) skew symmetric determinant of third order has the value zero. 

- **P-2** : If any two rows (or columns) of a determinant be inter changed, the value of determinant is changed in sign only. e.g. 

\[ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ h & g & i \end{vmatrix} \]

- **P-3** : If a determinant has any two rows (or columns) identical, then its value is zero. e.g. 

\[ \begin{vmatrix} a & b & c \\ a & b & c \\ p & q & r \end{vmatrix} = 0 \]

- **P-4** : If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number. e.g. 

\[ \begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ 3a & 3b & 3c \end{vmatrix} = 6 \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} \]

- **P-5** : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. e.g. 

\[ \begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 + x & b_2 + y & c_2 + z \\ a_3 + x & b_3 + y & c_3 + z \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ x & y & z \\ x & y & z \end{vmatrix} \]

- **P-6** : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column). e.g. 

\[ \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \]

**Note** : that while applying this property ATLEAST ONE ROW (OR COLUMN) must remain unchanged. **P-7** : If by putting \( x = a \) the value of a determinant vanishes then \( (x - a) \) is a factor of the determinant.

7. **Multiplication Of Two Determinants**

\[ \begin{vmatrix} a_1 & a_2 & \ldots \ldots & a_n \\ a_{11} & a_{12} & \ldots \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots \ldots & a_{mn} \end{vmatrix} \]

Similarly two determinants of order three are multiplied.

\[ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \]

**Note** : \( a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 = 0 \) etc. therefore 

**Proof** : Consider 

\[ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \]

8. **System Of Linear Equation (In Two Variables)**

(i) **Consistent Equations** : Definite & unique solution [ intersecting lines ]

(ii) **Inconsistent Equation** : No solution [ Parallel line ]

(iii) **Dependent equation** : Infinite solutions [ Identical lines ]

**Cramer's Rule** : [Simultaneous Equations Involving Three Unknowns]

\[ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \]

Let \( a \) \( x \) + \( b \) \( y \) + \( c \) \( z \) = \( d \) \( \ldots \ldots \) (I) \; \( a \) \( x \) + \( b \) \( y \) + \( c \) \( z \) = \( d \) \( \ldots \ldots \) (II) \; \( a \) \( x \) + \( b \) \( y \) + \( c \) \( z \) = \( d \) \( \ldots \ldots \) (III)

Then , 

\[ x = \frac{D_1}{D} , \quad Y = \frac{D_2}{D} , \quad Z = \frac{D_3}{D} \]

9. **6. Matrices**

**Useful In Study Of Science, Economics And Engineering**

1. **Definition** : Rectangular array of \( m \times n \) numbers. Unlike determinants it has no value.

\[ A = \begin{vmatrix} a_{11} & a_{12} & \ldots \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots \ldots & a_{mn} \end{vmatrix} \]

or

\[ A = \begin{vmatrix} a_{11} & a_{12} & \ldots \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots \ldots & a_{mn} \end{vmatrix} \]

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2. Special Type Of Matrices :
(a) Row Matrix : \( A = [a_{11}, a_{12}, \ldots, a_{1n}] \) having one row. (1 \times n) matrix. (or row vectors)
(b) Column Matrix : \( A = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix} \) having one column. (m \times 1) matrix (or column vectors)
(c) Zero or Null Matrix : \( (A = O_{m \times n}) \) An \( m \times n \) matrix all whose entries are zero.

A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} is a \( 3 \times 2 \) null matrix & \( B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) is \( 3 \times 3 \) null matrix

(d) Horizontal Matrix : A matrix of order \( m \times n \) is a horizontal matrix if \( n > m \).

\( \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix} \)

(e) Vertical Matrix : A matrix of order \( m \times n \) is a vertical matrix if \( m > n \).

\( \begin{bmatrix} 2 \\ 5 \\ 1 \\ 1 \\ 3 \\ 6 \\ 2 \\ 4 \end{bmatrix} \)

(f) Square Matrix : (Order \( n \)) If number of row = number of column \( \Rightarrow \) a square matrix.

\( \text{Note :} \) In a square matrix the pair of elements \( a_{ij} \) & \( a_{ji} \) are called \textbf{Conjugate Elements}. e.g. \( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \)

The elements \( a_{11}, a_{21}, a_{12}, a_{22} \) are called \textbf{Diagonal Elements}. The line along which the diagonal elements lie is called \textit{Principal or Leading} diagonal. The qty \( \Sigma a_{ii} \) is \textit{trace} of the matrix written as, \( \text{i.e. } \Sigma a_{ii} \) is the diagonal sum of the \textit{Principal or Leading} diagonal matrix.

\( \text{Note :} \) Min. number of zeros in a diagonal matrix of order \( n = \binom{n}{2} \) "It is to be noted that with square matrices there is a corresponding determinant formed by the elements of \( A \) in the same order."

3. \textbf{Equality Of Matrices :} Let \( A = [a_{ij}] \) & \( B = [b_{ij}] \) be equal if,

(i) Both have the same order. (ii) \( a_{ij} = b_{ij} \) for each \( i \& j \).

\( \text{4. Algebra Of Matrices :} \)
(a) \text{Addition of matrices is commutative.} i.e. \( A + B = B + A \), \( A, B = m \times n \), \( B = m \times n \)
(b) Matrix addition is associative, \( (A + B) + C = A + (B + C) \) Note : \( A, B, C \) are of the same type.
(c) Additive inverse. If \( A + B = O \) then \( A = -B \) & \( B = -A \).

5. \textbf{Multiplication Of A Matrix By A Scalar} : If \( A = [a_{ij}] \) & \( k \) is a scalar then \( kA = [ka_{ij}] \).

6. \textbf{Multiplication Of Matrices : (Row by Column)} \( AB \) exists if, \( A = m \times n \) & \( B = n \times p \). \( 2 \times 3 \) \( 3 \times 3 \)

\( AB \) exists, but \( BA \) does not \( \Rightarrow BA \neq AB \)

\( \text{Note :} \) In the product \( AB \), \( \frac{A}{B} \) is \( \text{pre factor} \) & \( \frac{B}{A} \) is \( \text{post factor} \).

\( A = [a_{ij}] \) \& \( B = [b_{ij}] \)

\( AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} \\ \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1} \end{bmatrix} \)

\( \text{Note :} \) In the product \( AB \), \( \frac{A}{B} = [a_{ij}] \) \& \( \frac{B}{A} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \)

\( B = [b_{ij}] \)

\( \text{Properties Of Matrix Multiplication :} \)

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(3) If \(|A| \neq 0\) & (adj A) . B = O (Null matrix), system is consistent having non-trivial solution

(4) If \(|A| = 0\), matrix method fails

7. LOGARITHM AND THEIR PROPERTIES

**Things To Remember:**

1. **LOGARITHM OF A NUMBER:**
   - The logarithm of the number \(N\) to the base \(a\) is the exponent indicating the power to which the base \(a\) must be raised to obtain the number \(N\).
   - Logarithm is an inverse function of exponential function.

2. **THE PRINCIPAL PROPERTIES OF LOGARITHMS:**
   - If \(M\) & \(N\) are arbitrary positive numbers,
     \(a > 0\), \(a \neq 1\) & \(N > 0\) This is known as the **Fundamental Logarithmic Identity**
     - \(\log_{a}N = N^{1/a} = e^{\ln N}\)

3. **PROPERTIES OF MONOTONOCITY OF LOGARITHMIC:**
   - For a > 1, the inequality \(0 < x < y\) & \(\log_{a}x < \log_{a}y\) are equivalent.
   - For \(0 < a < 1\) the inequality \(0 < x < y\) & \(\log_{a}x > \log_{a}y\) are equivalent.
   - If \(a > 1\) then \(\log_{a}x < p\) \(\Rightarrow 0 < x < a^{p}\)
   - If \(a > 1\) then \(\log_{a}x > p\) \(\Rightarrow x > a^{p}\)
   - If \(0 < a < 1\) then \(\log_{a}x < p\) \(\Rightarrow x < a^{p}\)

8. **PROBABILITY**

(i) **SAMPLE-SPACE:**
   - The set of all possible outcomes of an experiment is called the SAMPLE-SPACE (S).

(ii) **EVENT:**
   - A sub set of sample-space is called an EVENT.

(iii) **COMPLEMENT OF AN EVENT:**
   - The set of all out comes which are in S but not in A is called the COMPLEMENT OF THE EVENT A denoted by \(\overline{A}\) or \(A'\).

(iv) **COMPOUND EVENT:**
   - If A & B are two given events then A \(\cap\) B is called COMPOUND EVENT and is denoted by \(A \cap B\) or \(AB\) or \(A \& B\).

(v) **MUTUALLY EXCLUSIVE EVENTS:**
   - Two events are said to be MUTUALLY EXCLUSIVE (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other.
   - A & B are two mutually exclusive events then \(P(A \cap B) = 0\).
Events are said to be **equally likely** when each event is equally likely to occur as any other event.

**Examples:** Events A, B, C, .... L are said to be **equally likely** if no event outside this set can result as an outcome of an experiment. For example, if A & B are two events defined on a sample space S, then A & B are exhaustive if \( A \cup B = S \Rightarrow P(A \cup B) = 1 \).

**Classical Definition of Probability:** If \( n \) represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and in of them are favorable to the happening of the event \( A \), then the probability of happening of the event \( A \) is given by

\[
P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}} = \frac{n}{N}
\]

**Note:**
1. \( 0 \leq P(A) \leq 1 \)
2. \( P(A) + P(\overline{A}) = 1 \)

**Comparative Study of Equally Likely, Mutually Exclusive and Exhaustive Events.**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Events</th>
<th>E/L</th>
<th>M/E</th>
<th>Exhaustive</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Throwing a die</td>
<td>A: throwing an odd face {1, 3, 5}</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>B: throwing a composite face {4, 6}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. A ball is drawn from an urn containing 2W, 3R and 4G balls</td>
<td>( E_1 ): getting a W ball</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( E_2 ): getting a R ball</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_3 ): getting a G ball</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Throwing a pair of dice</td>
<td>( A ) : throwing a doublet {11, 22, 33, 44, 55, 66}</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>( B ) : throwing a total of 10 or more {46, 64, 55, 56, 65, 66}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. From a well shuffled pack of cards a card is drawn</td>
<td>( E_1 ): getting a heart</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( E_2 ): getting a spade</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_3 ): getting a diamond</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. From a well shuffled pack of cards a card is drawn</td>
<td>A = getting a heart</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>B = getting a face card</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Result 3:**

For any three events A, B and C we have (See Fig. 2)

\[
\begin{align*}
(P(A \cup B) + P(C) - P(A \cap C) - P(B \cap C)) & = 1 - P(A \cap B) - P(C) + P(A \cap C) + P(B \cap C) \\
(P(A \cup B) + P(C) - P(B \cap C)) & = 1 - P(A \cap B) - P(C) + P(A \cap C) + P(B \cap C)
\end{align*}
\]

**Note:** If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive. I.e. \( P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cup B \cap C) = 0 \). However, the converse of this is not true.

**Result 4: Independent Events**

Two events A & B are said to be independent if occurrence or non-occurrence of one does not affect the probability of occurrence or non-occurrence of other.

(i) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be **dependent or contingent**. For two independent events A and B, \( P(A \cap B) = P(A) \cdot P(B) \). Often this is taken as the definition of independent events.

(ii) Three events A, B & C are independent if & only if all the following conditions hold:

\[
P(A \cap B) = P(A) \cdot P(B) \\
P(A \cap C) = P(A) \cdot P(C) \\
P(B \cap C) = P(B) \cdot P(C) \\
P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)
\]

**Note:** Independent events are not in general mutually exclusive & vice versa.

**Result 5: Bayes’ Theorem or Total Probability Theorem.**

If an event can occur only with one of the \( n \) mutually exclusive and exhaustive events \( B_1, B_2, ..., B_n \), & the probabilities \( P(A/B_1), P(A/B_2), ..., P(A/B_n) \) are known then,

\[
P(B/A) = \frac{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}{P(A)}
\]

**Proof:** The events A occurs with one of the \( n \) mutually exclusive & exhaustive events \( B_1, B_2, ..., B_n \), \( A = B_1 \cup B_2 \cup ... \cup B_n \). Let \( P(B_i) \) be the probability of occurrence of alternative event \( B_i \). Then

\[
P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)
\]

**Note:** A event is the one for which

\[
P(B_i) = \text{event what we want} \quad B_1, B_2, ..., B_n \text{ are alternative event}
\]

Now,

\[
P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + ... + P(B_n) \cdot P(A/B_n)
\]

\[
P(B_1) \cdot P(B_2) \cdot ... \cdot P(B_n) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)
\]

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\[ P(B/A) = \frac{P(B) \cdot P(A/B)}{\sum P(B) \cdot P(A/B)} \]

**RESULT** 6 If \( p_1 \) and \( p_2 \) are the probabilities of speaking the truth of two independent witnesses A and B then \( P \) (their combined statement is true) = \( \frac{p_1 \cdot p_2}{p_1 \cdot p_2 + (1-p_1)(1-p_2)} \). In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

However if \( p \) is the probability of the happening of the event before their statement then \( P \) (their combined statement is true) = \( \frac{p_1 \cdot p_2}{p_1 \cdot p_2 - (1-p_1)(1-p_2)} \).

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.

If this is not the case and c is the chance of their coincidence then the probability that the statement is true = \( p_1 \cdot p_2 \cdot P \), that the statement is false = \( (1-p_1)(1-p_2)P \).

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

**RESULT** 7 (i) **Probability Distribution** spells out how a total probability of 1 is distributed over several values of a random variable. www.MathsBySuhag.com, www.TekoClasses.com

(ii) Mean of any probability distribution of a random variable is given by: \( \mu = \frac{\sum p \cdot x}{\sum p} \).

(iii) Variance of a random variable is given by: \( \sigma^2 = \sum (x - \mu)^2 \cdot p_i \).

(iv) The probability distribution for a binomial variate \( x \) is given by \( p(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} \).

(v) The variance of a binomial variate \( x \) is given by \( \sigma^2 = n \cdot p \cdot (1-p) \).

(vi) The standard deviation of a binomial variate \( x \) is given by \( \sigma = \sqrt{n \cdot p \cdot (1-p)} \).

(vii) The mean of a binomial variate \( x \) is given by \( \mu = np \).

**RESULTS 8 - B: GEOMETRICAL APPLICATIONS**

The following statements are axiomatic:

(i) If a point is taken at random on a given staright line AB, the chance that it falls on a particular segment PQ of the line is \( \overline{PQ}/AB \).

(ii) If a point is taken at random on the area S which includes an area \( \sigma \), the chance that the point falls on \( \sigma \) is \( \sigma/S \).

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**FUNCTIONS**

**9. FUNCTIONS**

**Things To Remember**
1. **General Definition**
   If to every value (considered as real unless otherwise stated) of a variable \( x \), which belongs to some collection (Set) \( E \), there corresponds one and only one finite value of the quantity \( y \), then \( y \) is said to be a function (Single valued) of \( x \) or a dependent variable defined on the set \( E \); \( x \) is the argument or independent variable.

2. If to every value of \( x \) belonging to some set \( E \) there corresponds one or several values of the variable \( y \), then \( y \) is called a multiple valued function of \( x \) defined on \( E \). Conventionally the word "FUNCTION" is used only as the meaning of a single valued function, if not otherwise stated. Pictorially: \( \frac{f(x)}{y} \) is the image of \( x \) and \( x \) is the pre-image of \( y \) under \( f \). Every function from \( A \to B \) satisfies the following conditions:

(i) \( f \subseteq A \times B \) (ii) \( \forall x \in A \Rightarrow (a, f(a)) \in f \) and (iii) \( (a, b) \in f \) & \( (a, c) \in f \Rightarrow b = c \)

2. **Domain, Co-Domain & Range of A Function**
   Let \( f: A \to B \), then the set \( A \) is known as the domain of \( f \) & the set \( B \) is known as co-domain of \( f \). The set of all \( f \) images of elements of \( A \) is known as the range of \( f \).

Domain of \( f = \{ a \mid a \in A, (a, f(a)) \in f \} \)

Range of \( f = \{ f(a) \mid a \in A, (a, f(a)) \in B \} \)

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives its domain.

3. **Important Types of Functions**
   (i) **Polynomial Function**
   If a function \( f \) is defined by \( f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \), where \( n \) is a non negative integer and \( a_0, a_1, a_2, \ldots, a_n \) are real numbers and \( a_1 \neq 0 \), then \( f \) is called a polynomial function of degree \( n \).

   **Note:** (a) A polynomial of degree one with constant term is called an odd linear function.

   (b) There are two polynomial functions, satisfying the relation: \( f(x) = f(1) + f(2) \).

   (ii) **Algebraic Function**
   If \( y \) is an algebraic function of \( x \), if it is a function that satisfies an algebraic equation of the form \( P_n(x) \cdot y^n + P_{n-1}(x) \cdot y^{n-1} + \ldots + P_1(x) \cdot y + P_0(x) = 0 \) where \( n \) is a positive integer and \( P_n(x), P_{n-1}(x), \ldots, P_0(x) \) are Polynomials in \( x \).

   Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called **transcendental function**.

   (iii) **Rational Function**
   A rational function is a function of the form: \( f(x) = \frac{g(x)}{h(x)} \), where \( g(x) \) & \( h(x) \) are polynomials & \( h(x) \neq 0 \).

   (iv) **Absolute Value Function**
   A function \( y = |x| \) is called the absolute value function or Modulus function. It is defined as: \( y = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases} \).

   (v) **Exponential Function**
   A function \( f(x) = e^{ax+b} (a > 0, a \neq 1, x \in R) \) is called an exponential function. The inverse of the exponential function is called the logarithmic function. \( g(x) = \log_x \). Note that \( f(x) \) & \( g(x) \) are inverse of each other & their graphs are as shown.

   (vi) **Signum Function**
   A function \( y = f(x) = \text{Sign}(x) \) is defined as follows:

   \[ y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \]

   It is also written as \( \text{Sign} x = \text{Sign} (x) ; x \neq 0 ; f(0) = 0 \).

   (vii) **Greatest Integer Or Step Up Function**
   The function \( y = f(x) = \lfloor x \rfloor \) is called the greatest integer function where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Note that:

   \[ -1 \leq x < 0 : \lfloor x \rfloor = -1 \quad 0 \leq x < 1 : \lfloor x \rfloor = 0 \]

   Properties of greatest integer function:

   (a) \( |x| \leq |x| < |x| + 1 \) and \( -x - 1 < |x| \leq 0, 0 < -x < |x| < 1 \)

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(b) \[ x + m = \{ x \} + m \text{ if } m \text{ is an integer} \]

(c) \[ |x| + |y| \leq |x + y| \leq |x| + |y| + 1 \]

(d) \[ |x + [-x]| = 0 \text{ if } x \text{ is an integer} \quad -1 \text{ otherwise} \]

(viii) **Fractional Part Function:**

It is defined as: \[ g(x) = \{ x \} = x - \lfloor x \rfloor \]

\( g(x) = \{ x \} \in \mathbb{R} \setminus \{0\} \cup \{0\} , \quad \text{if } n \text{ is odd} \]

\( \mathbb{R} \cup \{0\} , \quad \text{if } n \text{ is even} \)

\( \mathbb{R} , \quad \text{if } n \text{ is odd} \)

\( \mathbb{R} \setminus \{0\} \cup \{0\} , \quad \text{if } n \text{ is even} \)

4. **DOMAINS AND RANGES OF COMMON FUNCTION:**

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( x^n , \ n \in \mathbb{N} )</td>
<td>( \mathbb{R} ) (set of real numbers) ( \mathbb{R} ), ( n ) is odd</td>
<td>( \mathbb{R} \cup {0} ), ( n ) is even</td>
</tr>
<tr>
<td>(ii) ( \frac{1}{x^n} , \ n \in \mathbb{N} )</td>
<td>( \mathbb{R} \setminus {0} )</td>
<td>( \mathbb{R} \setminus {0} ), ( n ) is odd</td>
</tr>
<tr>
<td>(iii) ( x^{\frac{1}{n}} , \ n \in \mathbb{N} )</td>
<td>( \mathbb{R} ), ( n ) is odd</td>
<td>( \mathbb{R} ), ( n ) is even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mathbb{R} \cup {0} ), ( n ) is even</td>
</tr>
<tr>
<td>(iv) ( \frac{1}{x^{\pi n}} , \ n \in \mathbb{N} )</td>
<td>( \mathbb{R} \setminus {0} ), ( n ) is odd</td>
<td>( \mathbb{R} \setminus {0} ), ( n ) is even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mathbb{R} ), ( n ) is even</td>
</tr>
</tbody>
</table>

B. **Trigonometric Functions**

(i) \( \sin x \) \( \mathbb{R} \) \([-1, +1] \)

(ii) \( \cos x \) \( \mathbb{R} \) \([-1, +1] \)

(iii) \( \tan x \) \( \mathbb{R} \setminus \{2k+1\} \frac{\pi}{2}, k \in \mathbb{I} \) \( \mathbb{R} \)

(iv) \( \sec x \) \( \mathbb{R} \setminus \{2k+1\} \frac{\pi}{2}, k \in \mathbb{I} \) \((-\infty, -1] \cup [1, \infty) \)

(v) \( \csc x \) \( \mathbb{R} \setminus \{k\pi, k \in \mathbb{I} \} \) \((-\infty, -1] \cup [1, \infty) \)

(vi) \( \cot x \) \( \mathbb{R} \setminus \{k\pi, k \in \mathbb{I} \} \) \((-\infty, -1] \cup [1, \infty) \)

C. **Inverse Circular Functions** (Refer after Inverse is taught)

(i) \( \sin^{-1} x \) \([-1, +1] \)

(ii) \( \cos^{-1} x \) \([0, \pi] \)

(iii) \( \tan^{-1} x \) \( \mathbb{R} \)

(iv) \( \csc^{-1} x \) \((-\infty, -1] \cup [1, \infty) \)

(v) \( \sec^{-1} x \) \((-\infty, -1] \cup [1, \infty) \)

(vi) \( \cot^{-1} x \) \( \mathbb{R} \) \([0, \pi] \)

D. **Exponential Functions**

(i) \( e^x \) \( \mathbb{R} \) \( \mathbb{R}^+ \)

(ii) \( a^x , a > 0 \) \( \mathbb{R} \setminus \{0\} \) \( \mathbb{R}^+ \)

(iii) \( a^{\sin x} , a > 0 \) \( \mathbb{R} \setminus \{0\} \) \( \mathbb{R}^+ \)

E. **Logarithmic Functions**

(i) \( \log x , (a > 0 \ (a \neq 1) \) \( \mathbb{R}^+ \)

(ii) \( \log_a \frac{1}{x} , (a > 0 \ (a \neq 1) \) \( \mathbb{R}^+ \)

F. **Integral Part Functions**

(i) \( \lfloor x \rfloor \) \( \mathbb{R} \), \( n \in \mathbb{I} \)

(ii) \( \lfloor \frac{1}{x} \rfloor , (n \in \mathbb{I} \) \( \mathbb{R} \)

G. **Fractional Part Functions**

(i) \( \{ x \} \) \( \mathbb{R} \)

(ii) \( \{ \frac{1}{x} \} , (n \in \mathbb{I} \) \( \mathbb{R} \)

H. **Modulus Functions**

(i) \( | x | \) \( \mathbb{R} \)

(ii) \( | \frac{1}{x} | , (n \in \mathbb{I} \) \( \mathbb{R} \)

I. **Signum Function**

\( \text{sgn} (x) = \begin{cases} 1 & x \neq 0 \\ \frac{|x|}{x} & x = 0 \end{cases} \)

\( \mathbb{R} \setminus \{-1, 0, 1\} \)

J. **Constant Function**

\( f(x) = c \) \( \mathbb{R} \setminus \{c\} \)

5. **EQUAL OR IDENTICAL FUNCTION**

Two functions \( f \) & \( g \) are said to be equal if:

(i) \( \text{The domain of } f = \text{the domain of } g \)

(ii) \( \text{The range of } f = \text{the range of } g \)

(iii) \( f(x) = g(x) \) for every \( x \) belonging to their common domain. eg.

\( f(x) = \frac{1}{x} \) & \( g(x) = \frac{1}{x} \) are identical functions.

6. **CLASSIFICATION OF FUNCTIONS**

A function \( f : A \rightarrow B \) is said to be a one–one function if and only if different elements of \( A \) have different \( f \) images in \( B \). Thus for \( x_1 , x_2 \in A \) \& \( f(x_1) \neq f(x_2) \).

A function \( f : A \rightarrow B \) is said to be an onto function if \( f(A) = B \).

A function \( f : A \rightarrow B \) is said to be a one–one function if and only if \( f(x_1) \neq f(x_2) \) for every \( x_1 , x_2 \in A \) with \( x_1 \neq x_2 \).

Diagrammatically an injective mapping can be shown as:

Note: (i) Any function which is entirely increasing or decreasing in whole domain, then \( f(x) \) is one–one.

(ii) If any line parallel to \( x \)-axis cuts the graph of the function atmost at one point, then the function is one–one.

Many–one function:

A function \( f : A \rightarrow B \) is said to be a many one function if two or more elements of \( A \) have the same function value.
f image in B. Thus if \( f : A \to B \) is many one for \( x, x_1, x_2 \in A, f(x_1) = f(x_2) \) but \( x_1 \neq x_2 \).

Diagramatically a many one mapping can be shown as

\[
\begin{array}{c}
\bullet \quad x_1 \\
\bullet \quad x_2
\end{array}
\quad \text{A} \quad \text{B}
\]

Note : (i) Any continuous function which has at least one local maximum or local minimum, then \( f(x) \) is many one. In other words, if a line parallel to x-axis cuts the graph of the function at least at two points, then \( f \) is many one .

7. ALGEBRAIC OPERATIONS ON FUNCTIONS:

(i) \((f \pm g)\) \( (x) = f(x) \pm g(x) \) domain in each case is \( A \cap B \)

(ii) \((f, g)\) \( (x) = f(x) \cdot g(x) \)

(iii) \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \) domain is \( \{ x \mid x \in A \cap B \text{ s.t. } g(x) \neq 0 \} \).

8. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS:

Let \( f : A \to B \) & \( g : B \to C \) be two functions. Then the function \( (gof) (x) \) defined by \( (gof) (x) = g(f(x)) \) \( \forall x \in A \) is called the composite of the two functions \( f \) & \( g \). Diagonomically \( x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) \). Thus the image of every \( x \in A \) under the function \( g \) is the image of the \( f \)-image of \( x \).

Note that \( gof \) is defined only if \( \forall x \in A, f(x) \) is an element of the domain of \( g \) so that we can take its \( g \)-image. Hence for the product \( (gof) \) of two functions \( f \) & \( g \), the range of \( f \) must be a subset of the domain of \( g \).

Properties of Composite Functions:

(i) The composite of functions is not commutative i.e. \( f \circ g \neq g \circ f \).

(ii) The composite of functions is associative i.e. if \( f, g, h \) are three functions such that \( f \circ (g \circ h) \) & \( (f \circ g) \circ h \) are defined, then \( f \circ (g \circ h) = (f \circ g) \circ h \).

(iii) The composite of two bijections is a bijection i.e. if \( f \) & \( g \) are two bijections such that \( g \circ f \) is defined, then \( g \circ f \) is also a bijection.

9. HOMOGENEOUS FUNCTIONS:

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables. For example \( 5x^3 + 3y^2 - yx \) is homogeneous in \( x \) & \( y \). Symmetrically if \( f(tx, ty) = t^nf(x, y) \) then \( f(x, y) \) is homogeneous function of degree \( n \).

10. BOUNDED FUNCTION:

A function is said to be bounded if \( |f(x)| \leq M \), where \( M \) is a finite quantity.

11. IMPPLICIT & EXPLICIT FUNCTION:

A function defined by an equation not solved for the dependent variable is called an Implicit Function. For eg, the equation \( x^2 + y^2 = 1 \) defines \( y \) as an implicit function. If \( y \) has been expressed in terms of \( x \) alone then it is called an Explicit Function.

12. INVERSE OF A FUNCTION:

Let \( f : A \to B \) be a one-one & onto function, then there exists a unique function \( g : B \to A \) such that \( f(x) = y = g(y) = x, \forall x \in A \& y \in B \). Then \( g \) is said to be inverse of \( f \). Thus \( g = f^{-1} : B \to A = \{ (f(x), x) \mid x \in f \} \).

Properties of Inverse Function:

(i) The inverse of a bijection is unique.

(ii) If \( f : A \to B \) is a bijection & \( g : B \to A \) is the inverse of \( f \), then \( fog = I_B \) & \( gof = I_A \), where \( I_A \) & \( I_B \) are identity functions on the sets \( A \) & \( B \) respectively.

Note that the graphs of \( f \) & \( g \) are the mirror images of each other in the line \( y = x \). As shown in the figure given below a point \( (x, y) \) corresponding to \( y = x^2 (x \geq 0) \) changes to \( (y, x) \) corresponding to \( y = \sqrt{x} \), the changed form of \( x = \sqrt{y} \).

13. ODD & EVEN FUNCTIONS:

(i) \((f \circ g) (x) = f(g(x)) \) for all \( x \in \text{domain of } f \) then \( f \) is said to be an even function. e.g. \( f(x) = x^2 \) \( \forall x \in \mathbb{R} \) & \( g(x) = x^2 + 3 \). If \( f(x) = f(-x) \) for all \( x \) in the domain of \( f \) then \( f \) is said to be an odd function. e.g. \( f(x) = \sin x \) \( \forall x \in \mathbb{R} \).

Note : (a) \( f(x) = f(-x) \), \( f(x) \) is even & \( f(x) \) is odd.

(b) A function may either be odd or even. (c) Inverse of an even function is not defined

(d) Every even function is symmetric about the \( y \)-axis & every odd function is symmetric about the origin.
14. PERIODIC FUNCTION: A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of x e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

Note: (a) f (T) = f (0) = f (−T), where T is the period.
(c) Every constant function is always periodic, with no fundamental period.

15. GENERAL: If x, y are independent variables, then:
(i) f(xy) = f(x) . f(y) ⇒ f(x) = k ln x or f(x) = x .
(ii) f(xy) = f(x) + f(y) ⇒ f(x) = x^n, n ∈ R
(iii) f(x + y) = f(x) + f(y) ⇒ f(x) = a^x.
(iv) f(x) = f(ax + b) has a period T/a.

10. INVERSE TRIGONOMETRY FUNCTION

GENERAL DEFINITION(S): sin^−1 x, cos^−1 x, tan^−1 x etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arcsin x, arccos x etc.

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS:
- (i) y = sin^−1 x where −1 ≤ x ≤ 1; −π/2 ≤ y ≤ π/2 and sin y = x.
- (ii) y = cos^−1 x where −1 ≤ x ≤ 1; 0 ≤ y ≤ π and cos y = x.
- (iii) y = tan^−1 x where x ∈ R; −π/2 < x < 1/2 and tan y = x.
- (iv) y = cosec^−1 x where x ≤ −1 or x ≥ 1; −π/2 ≤ y ≤ π/2, y ≠ 0 and cosec y = x.
- (v) y = sec^−1 x where x ≤ −1 or x ≥ 1; 0 ≤ y ≤ π; y ≠ π/2 and sec y = x.
- (vi) y = cot^−1 x where x ∈ R, 0 < y < π and cot y = x.

Note: (a) 1st quadrant is common to all the inverse functions.
(b) 3rd quadrant is not used in inverse functions.
(c) 4th quadrant is used in the CW.[W] DIRECTION i.e. −π/2 ≤ y ≤ 0.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS:
   P−1
   (i) sin(−1)x = x, −1 ≤ x ≤ 1
   (ii) cos(−1)x = x, −1 ≤ x ≤ 1
   (iii) tan(−1)x = x, x ∈ R
   (iv) sin^−1(sinx) = x, −π/2 ≤ x ≤ π/2
   (v) cos^−1(cosx) = x, 0 ≤ x ≤ π
   (vi) tan^−1(tanx) = x, −π/2 < x < π/2

   P−2
   (i) sec(−1)x = cos(−1)x, −1 ≤ x ≤ 1
   (ii) cos(−1)x = −sec(−1)x, 1 ≤ x ≤ +∞
   (iii) tan(−1)x = sec(−1)x, −1 ≤ x ≤ 1
   (iv) cos^−1(x) = π − sin^−1 x, −1 ≤ x ≤ 1
   (v) sin^−1(x) = cos^−1(−x), −1 ≤ x ≤ 1
   (vi) tan^−1(x) = −cot^−1(−x), 1 ≤ x ≤ +∞

   P−3
   (i) sin^−1(−x) = −sin^−1 x, −1 ≤ x ≤ 1
   (ii) tan^−1(−x) = −tan^−1 x, x ∈ R
   (iii) sin^−1(x) = π − cos^−1 x, −1 ≤ x ≤ 1
   (iv) cot^−1(−x) = tan^−1 x, x ∈ R

   P−4
   (i) sin^−1(x) + cos^−1(x) = π/2, x ∈ [−1,1]
   (ii) tan^−1(x) + cot^−1(x) = π/2, x ∈ R

   P−5
   tan^−1(x) + tan^−1(y) = tan^−1 \left( \frac{x+y}{1-xy} \right), x > 0, y > 0 & xy < 1

   P−6
   (i) sin^−1 x + sin^−1 y = sin^−1 \left[ x\sqrt{1-y^2} + y\sqrt{1-x^2} \right], x ≥ 0, y ≥ 0 & (x^2 + y^2) ≤ 1
   (ii) cos^−1 x + cos^−1 y = cos^−1 \left[ 1-2xy \right], x ≥ 0, y ≥ 0 & (x^2 + y^2) ≤ 1

   P−7
   (i) If tan^−1 x + tan^−1 y = tan^−1 \left( \frac{x+y}{1-xy} \right), x > 0, y > 0 & xy < 1

   P−8
   (i) 2 tan^−1 x = sin^−1 \left( \frac{2x}{1+x^2} \right) = cos^−1 \left( \frac{1-x^2}{1+x^2} \right) = tan^−1 \left( \frac{2x}{1-x^2} \right)

   Note carefully that:
   sin^−1 \frac{2x}{1+x^2} = \begin{cases} \frac{2\tan^−1 x}{\pi-2\tan^−1 x} & \text{if } \frac{x}{1+x^2} \leq 1 \\
\pi-2\tan^−1 x & \text{if } x > 1
\end{cases}

\cos^−1 \frac{1-x^2}{1+x^2} = \begin{cases} \frac{2\tan^−1 x}{\pi+2\tan^−1 x} & \text{if } x = 0 \\
\frac{2\tan^−1 x}{\pi-2\tan^−1 x} & \text{if } x < 0
\end{cases}

\tan^−1 \frac{2x}{1-x^2} = \begin{cases} \frac{2\tan^−1 x}{\pi+2\tan^−1 x} & \text{if } \frac{x}{1+x^2} < 1 \\
\frac{2\tan^−1 x}{\pi-2\tan^−1 x} & \text{if } x > 1
\end{cases}

Remember that:
   (i) sin^−1 x + sin^−1 y + sin^−1 z = \frac{3\pi}{2} \Rightarrow x = y = z = 1
   (ii) cos^−1 x + cos^−1 y + cos^−1 z = 3π \Rightarrow x = y = z = 1
   (iii) tan^−1 x + tan^−1 y + tan^−1 z = π \Rightarrow tan^−1 x + tan^−1 y + tan^−1 z = \frac{\pi}{2}
9. (a) \( y = \tan^{-1}(\tan x), \ x \in \mathbb{R} \), \( y \) is aperiodic

9. (b) \( y = \tan(\tan^{-1}x), \ y \in \mathbb{R} \), periodic with period \( \pi \)

10. (a) \( y = \cot^{-1}(\cot x), \ y \in (0, \pi) \), periodic with period \( \pi \)

10. (b) \( y = \cot(\cot^{-1}x), \ y \in \mathbb{R} \), \( y \) is aperiodic

11. (a) \( y = \cosec^{-1}(\cosec x), \ y \in \left( -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right) \), periodic with period \( 2\pi \)

11. (b) \( y = \cosec(\cosec^{-1}x), \ y \in \mathbb{R} \), \( y \) is aperiodic

12. (a) \( y = \sec^{-1}(\sec x), \ y \in \mathbb{R} \), periodic with period \( 2\pi \)

12. (b) \( y = \sec(\sec^{-1}x), \ y \in \left( -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right) \), periodic with period \( 2\pi \)


11. Limit and Continuity

& Differentiability of Function

Things To Remember:

1. Limit of a function \( f(x) \) is said to exist as, \( x \to a \) when \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \text{finite quantity.} \)

2. Fundamental Theorems On Limits:

   Let \( \lim_{x \to a} f(x) = l \) & \( \lim_{g(x)} g(x) = m \). If \( l \) & \( m \) exists then:

   (i) \( \lim_{x \to a} [f(x) \pm g(x)] = l \pm m \)

   (ii) \( \lim_{x \to a} [f(x) \times g(x)] = l \times m \)

   (iii) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m} \), provided \( m \neq 0 \).

   (iv) \( \lim_{x \to a} k f(x) = k \lim_{x \to a} f(x) \); where \( k \) is a constant.

   (v) \( \lim_{x \to a} f[g(x)] = \left( \lim_{x \to a} g(x) \right) = f(m) \); provided \( f \) is continuous at \( g(x) = m \).

   For example, \( \lim_{x \to a} \ln f(x) = \ln \left( \lim_{x \to a} f(x) \right) \) if \( l(\rightarrow > 0) \).

3. Standard Limits:

   (a) \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

   (b) \( \lim_{x \to 0} \frac{1 + x}{1 - x} = \frac{1}{x} \)

   (c) If \( \lim_{x \to a} f(x) = l \) & \( \lim_{x \to a} \phi(x) = \infty \), then \( \lim_{x \to a} \psi(x) = \infty \).

   (d) \( \lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n \).

   (e) \( \lim_{x \to a} x^n = a^n \).

4. Squeeze Play Theorem:

   If \( f(x) \leq g(x) \leq h(x) \) \( \forall x \) & \( \lim_{x \to a} f(x) = l = \lim_{x \to a} h(x) \) then \( \lim_{x \to a} g(x) = l \).

5. Indeterminate Forms:

   \( \frac{0}{0} \), \( \frac{\infty}{\infty} \), \( 0^0 \), \( \infty^0 \), \( \infty \times 0 \), \( 0^\infty \), \( \infty - \infty \), \( 0^0 \) & \( 1^\infty \).

   Note:

   (i) We cannot plot \( \infty \) on the paper. Infinity \((\infty)\) is a symbol & not a number. It does not obey the laws of
   elementary algebra.

   (ii) \( a^\infty \) or \( \infty^a \) or \( \infty \times 0 \) or \( 0^\infty \) or \( \infty - \infty \)

   (iv) \( a^\infty = 0 \) if \( a \) is finite

   (v) \( a^\infty = 0 \) if \( a \) is not defined. 

6. The following strategies should be born in mind for evaluating the limits:

   (a) Factorisation

   (b) Rationalisation or double rationalisation

   (c) Use of trigonometric transformation;

   (d) Appropriate substitution and using standard limits

   (e) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of \( \sin x \), \( \cos x \), \( \tan x \) should be remembered by heart & are given below:

   (i) \( a^x = 1 + x \ln a + \frac{x^2}{2!} (\ln a)^2 + \frac{x^3}{3!} (\ln a)^3 + \cdots \)

   (ii) \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \)

   (iii) \( \ln (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{for} -1 < x \leq 1 \).

   (iv) \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \)

   (v) \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \)

   (vi) \( \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \).

   (vii) \( \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \)

   (viii) \( \sin^{-1} x = \frac{x^3}{3} + \frac{x^5}{15} + \frac{2x^7}{105} + \cdots \)

   (ix) \( \sec^{-1} x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots \)

   (CONTINUITY)

Things To Remember:

1. A function \( f(x) \) is said to be continuous at \( x = c \) if \( \lim_{x \to c} f(x) = f(c) \). Symbolically \( f \) is continuous at \( x = c \) if \( \lim_{x \to c} f(c - h) = \lim_{x \to c} f(c + h) = f(c) \).

   i.e. \( LH \) at \( x = c \) equals \( RH \) at \( x = c \).

   It should be noted that continuity of a function at \( x = a \) is meaningful only if the function is defined in the immediate neighbourhood of \( x = a \), not necessarily at \( x = a \).

2. Reasons of discontinuity:

   (a) \( \lim_{x \to a} f(x) \) does not exist

   i.e. \( \lim_{x \to a} f(x) \neq \lim_{x \to a} f(x) \)

   (ii) \( f(x) \) is not defined at \( x = c \).

   (iii) \( \lim_{x \to a} f(x) \neq f(c) \)

   Geometrically, the graph of the function will exhibit a break at \( x = c \).

   The graph shown is discontinuous at \( x = 1 \), 2 and 3.

3. Types of Discontinuities:

   Type-1 (Removable type of discontinuities):

   In case \( \lim_{x \to a} f(x) \) exists but is not equal to \( f(c) \) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that \( \lim_{x \to a} f(x) = f(c) \) & make it continuous at \( x = c \). Removable type of discontinuity can be further classified as:

   (a) Missing Point Discontinuity:

   Where \( \lim_{x \to a} f(x) \) exists finitely but \( f(a) \) is not defined.

   e.g. \( f(x) = \frac{(1-x)(9-x^2)}{(1-x)} \) has a missing point discontinuity at \( x = 1 \) & \( f(x) = \frac{\sin x}{x} \) has a missing point discontinuity at \( x = 0 \).

   (b) Isolated Point Discontinuity:

   Where \( \lim_{x \to a} f(x) \) exists & \( f(a) \) also exists but \( \lim_{x \to a} f(x) \neq f(a) \). e.g. \( f(x) = \frac{x^2 - 16}{x - 4} \), \( x \neq 4 \) & \( f(4) = 9 \) has an isolated point discontinuity at \( x = 4 \).

   Similarly \( f(x) = [x] + [-x] \)

   has an isolated point discontinuity at all \( x \in I \).

   Type-2: Non-Removable type of discontinuities:

   In case \( \lim_{x \to a} f(x) \) does not exist then it is not possible to make the function continuous by redefining it.

   Such discontinuities are known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as:

   (a) Finite discontinuity e.g. \( f(x) = x - \lfloor x \rfloor \) at all integral \( x \); \( f(x) = \tan^{-1} \frac{1}{x} \) at \( x = 0 \) and \( f(x) = \frac{1}{1 + 2^x} \) at \( x = 0 \) (note that \( f(0^+) = 0 \); \( f(0^-) = 1 \).)
(b) Infinite discontinuity e.g. \( f(x) = \frac{1}{x-4} \) or \( g(x) = \frac{1}{(x-4)^2} \) at \( x = 4 \); \( f(x) = 2^{\text{even}} \) at \( x = \frac{\pi}{2} \) and \( f(x) = \cos x \) at \( x = 0 \).

(c) Oscillatory discontinuity e.g. \( f(x) = \sin \frac{1}{x} \) at \( x = 0 \).

In all these cases the value of \( f(a) \) of the function at \( x = a \) (point of discontinuity) may or may not exist but \( \lim_{x \to a} f(x) \) does not exist.

Note: From the adjacent graph note that:
- \( f \) is continuous at \( x = -1 \)
- \( f \) has isolated discontinuity at \( x = 1 \)
- \( f \) has non removable (finite type) discontinuity at the origin.

4. In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at \( x = c \) & LHL at \( x = c \) is called the \textbf{Jump of Discontinuity}. A function having a finite number of jumps in a given interval \( I \) is called \textbf{piecewise continuous} or \textbf{sectionally continuous} function in this interval.

5. All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

6. If \( f \) & \( g \) are two functions that are continuous at \( x = c \) then the functions defined by:
\[
F(x) = f(x) \pm g(x) ; F(x) = K f(x), K \text{ any real number} ; F(x) = f(x).g(x)
\]
are also continuous at \( x = c \).

Further, if \( g(c) \) is not zero, then \( F(x) = \frac{f(x)}{g(x)} \) is also continuous at \( x = c \).

7. The \textbf{Intermediate Value Theorem}:
Suppose \( f(x) \) is continuous on an interval \( I \) and \( a \) and \( b \) are any two points of \( I \). Then if \( y_0 \) is a number between \( f(a) \) and \( f(b) \), then there exists a number \( c \) between \( a \) and \( b \) such that \( f(c) = y_0 \).

**NOTE:** Very carefully that \( f(x) \) is continuous in \( I \).

(a) If \( f(x) \) is continuous & \( g(x) \) is discontinuous at \( x = a \) then the function \( \phi(x) = f(x) \) \( g(x) \)

is not necessarily be discontinuous at \( x = a \). e.g. \( f(x) = x \) & \( g(x) = \left\{ \begin{array}{ll}
\sin \frac{\pi}{x} & x \neq 0 \\
0 & x = 0
\end{array} \right. \)

(b) If \( f(x) \) and \( g(x) \) both are discontinuous at \( x = a \) then the product function \( \phi(x) = f(x).g(x) \) is not necessarily be discontinuous at \( x = a \). e.g. \( f(x) = -g(x) = \left\{ \begin{array}{ll}
1 & x \geq 0 \\
-1 & x < 0
\end{array} \right. \)

Point functions are to be treated as discontinuous. e.g. \( f(x) = \sqrt{\frac{1}{x^2} - 1} \) is not continuous at \( x = 1 \).

(d) A Continuous function whose domain is closed must have a range also in closed interval.

If \( f(x) \) is continuous at \( x = c \) & \( g(x) \) is continuous at \( x = c \), then the composite \( g(f(x)) \) is continuous at \( x = c \). e.g. \( f(x) = \frac{x \sin x}{x^2 + 1} \) & \( g(x) = |x| \) are continuous at \( x = 0 \), hence the composite \( g(f(x)) = \frac{x \sin x}{x^2 + 1} \) will also be continuous at \( x = 0 \).\( \text{www.MathBySuhag.com} , \text{www.TekoClasses.com} \)

7. **Continuity In An Interval** :
(a) A function \( f \) is said to be continuous in \( (a, b) \) if \( f \) is continuous at each & every point \( \epsilon (a, b) \).

(b) A function \( f \) is said to be continuous in a closed interval \([a, b]\) if:
(i) \( f \) is continuous in the open interval \((a, b)\).
(ii) \( f \) is right continuous at \( a \)'s i.e. \( \lim_{x \to a^+} f(x) = f(a) \) is finite quantity.
(iii) \( f \) is left continuous at \( b \)'s i.e. \( \lim_{x \to b^-} f(x) = f(b) \) is finite quantity.

Note: If function \( f \) which is continuous in \([a, b]\) possesses the following properties:
(i) If \( f(a) \) & \( f(b) \) possess opposite signs, then there exists at least one solution of the equation \( f(x) = 0 \) in the open interval \((a, b)\).

(ii) If \( K \) is any real number between \( f(a) \) & \( f(b) \), then there exists at least one solution of the equation \( f(x) = K \) in the open interval \((a, b)\).

8. **Single Point Continuity**:
Functions which are continuous only at one point are said to exhibit single point continuity.

\[ ... n \text{ such that if } x \in \mathbb{Q} \text{ and } g(x) = \left\{ \begin{array}{ll}
x \text{ if } x \in \mathbb{Q} \\
0 \text{ if } x \notin \mathbb{Q}
\end{array} \right. \text{ are both continuous only at } x = 0. \]

**DIFFERENTIABILITY**

**Things To Remember :**
1. **Right hand & Left hand Derivatives**: By definition:
\[
f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}
\]
if it exist.

(i) The right hand derivative of \( f' \) at \( x = a \) denoted by \( f'(a^+) \) is defined by:
\[
f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}
\]
provided the limit exists & is finite.\( \text{www.MathBySuhag.com} , \text{www.TekoClasses.com} \)

(ii) The left hand derivative of \( f \) at \( x = a \) denoted by
\[
f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}
\]
from \( \text{www.MathBySuhag.com} , \text{www.TekoClasses.com} \)

(iii) Derivability & Continuity:
(a) If \( f'(a) \) exists then \( f(x) \) is derivable at \( x = a \) \( \Rightarrow \) \( f(x) \) is continuous at \( x = a \).

(b) If a function \( f \) is derivable at \( x = a \) then \( f \) is continuous at \( x = a \).

For: \( f'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \) exists.

Also \( f(x+h) - f(x) = \frac{f(x+h) - f(x)}{h} h \ [h \neq 0] \)

Therefore: \( f(x+h) - f(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} h = f'(x) h = 0 \)

Therefore \( \lim_{h \to 0^-} f(x+h) - f(x) = \lim_{h \to 0^-} f(x+h) = f(x) \text{ is continuous at } x \).

**Note**: If \( f(x) \) is derivable for every point of its domain of definition, then it is continuous in that domain.

The converse of the above result is not true.

* IF \( f(x) \) IS CONTINUOUS AT \( x \), THEN \( f(x) \) IS DERIVABLE AT \( x \) IS NOT TRUE.

e.g. The functions \( f(x) = |x| \) & \( g(x) = x \sin \frac{1}{x} \) are both continuous at \( x = 0 \) but not derivable at \( x = 0 \).

**NOTE CAREFULLY**:
(a) Let \( f'(a) = p \) & \( f'(a) = q \) where \( p \) & \( q \) are finite then:
(i) \( p = q \text{ if } f \) is derivable at \( x = a \) \( \Rightarrow \) \( f \) is continuous at \( x = a \).
(ii) \( p \neq q \text{ if } f \) is not derivable at \( x = a \).

It is very important to note that \( f \) may be still continuous at \( x = a \).

In short, for a function \( f \):

Differentiability \( \Rightarrow \) Continuity ; Continuity \( \not\Rightarrow \) derivability ;
Non derivability⇒ discontinuous but discontinuity ⇒ Non derivability

(b) If a function f is not differentiable but is continuous at x = a if geometrically implies a sharp corner at x = a.

3. **DERIVATIVITY OVER AN INTERVAL**:
   f(x) is said to be derivable over an interval if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if it:
   (i) for the points a and b, f′(a+) & f′(b−) exist &
   (ii) for any point c such that a < c < b, f′(c−) & f′(c−) exist & are equal.

**Note**: If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) − g(x), f(x)g(x) will also be derivable at x = a & if g(a) ≠ 0 then the function f(x)/g(x) will also be derivable at x = a.

3. **DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE / INITIO METHOD**:

2. A surprising result:

5. **DERIVABILITY**

6. **Inverse functions and their derivatives**:

7. **Logarithmic differentiation**:

8. **Implicit differentiation**:

9. **Parametric differentiation**:

10. **Derivative of a function W.R.T. another function**:

11. **Derivatives of order two & three**:

12. **Differentiation & l'Hospital Rule**

1. **Definition**:

5. **Derivative of standards functions**:

6. **Inverse functions and their derivatives**:

7. **Derivative of a function W.R.T. another function**:

8. **Logarithmic differentiation**:

9. **Implicit differentiation**:

10. **Parametric differentiation**:

11. **Derivatives of order two & three**:

12. **Differentiation & l'Hospital Rule**

1. **Definition**:

If x and x+h belong to the domain of a function f defined by y = f(x), then

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

if it exists, is called the **Derivative** of f at x and is denoted by f′(x) or \( \frac{dy}{dx} \). We have therefore, f′(x) = \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

2. The derivative of a given function f at a point x = a of its domain is defined as:

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

provided the limit exists & is denoted by f′(a).

Note that alternatively, we can define f′(a) = \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \), provided the limit exists.

3. **Derivative of (f(x)) from the first principle / initio method**:

If f(x) is a derivable function then, \( \lim_{h \to 0} \frac{\delta y}{\delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f′(x) = \frac{dy}{dx} \).

4. **Theorems on derivatives**:

If u and v are derivable function of x, then,

(i) \( \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \)

(ii) \( \frac{d}{dx}(Ku) = K \frac{du}{dx} \), where K is any constant

(iii) \( \frac{d}{dx}(uv) = u \frac{dv}{dx} \pm v \frac{du}{dx} \) known as “Product Rule”

(iv) \( \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \) where v ≠ 0 known as “Quotient rule”

(v) If y = f(u) & u = g(x) then \( \frac{dy}{dx} = \frac{du}{dx} \frac{dy}{du} \) “Chain rule”

3. **Derivative of standards functions**:

(i) D (x^n) = nx^{n-1} & g(x) = (x^n)

(ii) D (e^x) = e^x

(iii) D (a^x) = a^x \ln a \quad a > 0

(iv) D (ln x) = \frac{1}{x}

(v) D (log b x) = \frac{1}{x} \cdot \frac{1}{\ln b}

(vi) D (sin x) = cos x

(vii) D (cos x) = -sin x

(viii) D (tan x) = sec^2 x

(ix) D (sec x) = sec x . tan x

(x) D (cosec x) = -cosec x . cot x

(xi) D (cot x) = -cosec^2 x

(xii) D (constant) = 0

6. **Inverse functions and their derivatives**:

(a) **Theorem**:

If the inverse functions f & g are defined by y = f(x) & x = g(y) & if f′(x) exists & f′(x) ≠ 0 then g′(y) = \( \frac{1}{f′(x)} \). This result can also be written as, if f′(x) exists & \( \frac{dy}{dx} \neq 0 \), then

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}
\]

(b) **Results**:

(i) D(sin^(-1) x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1

(ii) D(cos^(-1) x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1

(iii) D(tan^(-1) x) = \frac{1}{1+x^2}, \quad x \in R

(iv) D(sec^(-1) x) = \frac{1}{|x| \sqrt{x^2-1}}, \quad |x| > 1

(v) D(cosec^(-1) x) = \frac{-1}{|x| \sqrt{x^2-1}}, \quad |x| > 1

Note: In general if y = f(u) then \( \frac{dy}{dx} = f′(u) \cdot \frac{du}{dx} \).

7. **Logarithmic differentiation**: To find the derivative of:

(i) a function which is the product or quotient of a number of functions OR

(ii) a function of the form \[ f(x)^{g(x)} \] where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate. This is called **Logarithmic Differentiation**.

8. **Implicit differentiation**: If \( \phi (x, y) = 0 \)

(i) In order to find \( \frac{dy}{dx} \) in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in \( \frac{dy}{dx} \) together on one side to finally find \( \frac{dy}{dx} \).

(ii) In answers of \( \frac{dy}{dx} \) in the case of implicit functions, both x & y are present.

9. **Parametric differentiation**: If y = f(x) & x = g(\theta) where \( \theta \) is a parameter, then \( \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \).

10. **Derivative of a function w.r.t. another function**: If y = f(x) then \( \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dx} \).

11. **Derivatives of order two & three**:

Let a function y = f(x) be defined on an open interval (a, b). It’s derivative, if it exists on (a, b) is a certain function f′(x) or (dy/dx) or y′ & is called the first derivative of y w.r.t. x.If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by f′′(x) or (d^2y/dx^2) or y′′. Similarly, the 3rd order
Consider \( y = f(x) \) and \( y = g(x) \) are functions of \( x \) such that:

(i) \( \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \) \ OR \ \( \lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x) \)

(ii) Both \( f(x) \) & \( g(x) \) are continuous at \( x = a \)

(iii) Both \( f(x) \) & \( g(x) \) are differentiable at \( x = a \)

(iv) Both \( f'(x) \) & \( g'(x) \) are continuous at \( x = a \), then

\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left( \frac{f'(x)}{g'(x)} \right) \quad \text{and} \quad \lim_{x \to a} \left( \frac{f''(x)}{g''(x)} \right)
\]

soon till indeterminate form vanishes.

14. **ANALYSIS AND GRAPHS OF SOME USEFUL FUNCTIONS:**

(i) \( y = f(x) = \sin^{-1} \left( \frac{2x}{1+x^2} \right) \)

\[
\begin{align*}
\text{if } & x \leq 1 \\
\text{if } & x > 1
\end{align*}
\]

\[
\begin{align*}
\text{if } & x \leq 1 \\
\text{if } & x > 1
\end{align*}
\]

\[
\begin{align*}
\text{if } & x \leq 1 \\
\text{if } & x > 1
\end{align*}
\]

\[
\begin{align*}
\text{if } & x \leq 1 \\
\text{if } & x > 1
\end{align*}
\]

HIGHLIGHTS:

(a) Domain is \( x \in \mathbb{R} \) & range is \( \left[ \frac{\pi}{2}, \frac{\pi}{2} \right] \)

(b) Not derivable at \( |x| = \frac{1}{2} \)

(c) \( \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \) if \( x \in \left( -\frac{1}{2}, \frac{1}{2} \right) \) & \( \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \) if \( x \in \left( -1, -\frac{1}{2} \right) \cup \left( \frac{1}{2}, 1 \right) \)

(d) Continuous everywhere in its domain

(ii) \( y = f(x) = \cos^{-1} \left( \frac{1-x}{1+x} \right) \)

\[
\begin{align*}
\text{if } & x \geq 0 \\
\text{if } & x < 0
\end{align*}
\]

\[
\begin{align*}
\text{if } & x \geq 0 \\
\text{if } & x < 0
\end{align*}
\]

\[
\begin{align*}
\text{if } & x \geq 0 \\
\text{if } & x < 0
\end{align*}
\]

HIGHLIGHTS:

(a) Domain is \( x \in \mathbb{R} \) & range is \( [0, \pi] \)

(b) Continuous for all \( x \) but not derivable at \( x = 0 \)

(c) \( \frac{dy}{dx} = \frac{2}{1+x^2} \) for \( x > 0 \) & \( \frac{dy}{dx} = \frac{-2}{1+x^2} \) for \( x < 0 \)

(d) \( \frac{dy}{dx} = 0 \) in \( (0, \infty) \) & \( \frac{dy}{dx} = \frac{2}{1+x^2} \) in \( (-\infty, 0) \)

GENERAL NOTE:

Concavity in each case is decided by the sign of 2nd derivative as:

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13. APPLICATION OF DERIVATIVE (AOD), TANGENT & NORMAL

THINGS TO REMEMBER :

I. The value of the derivative at P(x₁, y₁) gives the slope of the tangent to the curve at P. Symbolically
\[ f'(x₁) = \frac{dy}{dx} \bigg|_{x₁} = \text{Slope of tangent at } P(x₁, y₁) = m \text{ (say).} \]

II. Equation of tangent at \((x₁, y₁)\) is:
\[ y - y₁ = \frac{dy}{dx} \bigg|_{x₁} (x - x₁). \]

III. Equation of normal at \((x₁, y₁)\) is:
\[ y - y₁ = -\frac{1}{\frac{dy}{dx} \bigg|_{x₁}} (x - x₁). \]

1. The point \((x₁, y₁)\) will satisfy the equation of the curve & the equation of tangent & normal line.
2. If the tangent at any point P on the curve is // to the axis of x then dy/dx = 0 at the point P.
3. If the tangent at any point on the curve is parallel to the axis of y, then dx/dy = 0 = or dx/dy = 0.
4. If the tangent at any point on the curve is equally inclined to both the axes then dx/dy = ± 1.
5. If the tangent at any point makes equal intercept on the coordinate axes then dx/dy = ± 1.
6. If f is increasing in \([a, b]\) then f(x) is an increasing function at every point in the open interval a < x < b.
7. If a function is invertible it has to be either increasing or decreasing.
8. If f'(x) is every where negative, then f(x) is decreasing. If f'(x) is every where positive, then f(x) is increasing.

MONOTONICITY (Significance of the sign of the first order derivative)

DEFINITIONS :

1. A function f(x) is called an Increasing Function at a point x = a if in a sufficiently small neighbourhood around x = a we have
\[ f(x + h) > f(a) \]
and \( f(x - h) < f(a) \) increasing;

Similarly decreasing if
\[ f(x + h) < f(a) \]
and \( f(x - h) > f(a) \) decreasing.

2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined. www.MathsBySuhag.com, www.TekoClasses.com

3. A function which in a given interval is increasing or decreasing is called "Monotonic" in that interval.

4. Tests for increasing and decreasing of a function at a point:
   If the derivative \( f'(x) \) is positive at a point \( x = a \), then the function \( f(x) \) at this point is increasing. If it is negative, then the function is decreasing. Even if \( f'(a) \) is not defined, f can still be increasing or decreasing.

5. Tests for Increasing & Decreasing of a function in an interval:
   **SUFFICIENCY TEST**: If the derivative function \( f'(x) \) in an interval \((a, b)\) is every where positive, then the function \( f(x) \) in this interval is Increasing; www.MathsBySuhag.com, www.TekoClasses.com
   If \( f'(x) = 0 \) for every x in an interval \((a, b)\) then \( f(x) \) is a constant function in that interval.

6. (a) ROLLE'S THEOREM:
   Let \( f(x) \) be a function of \( x \) subject to the following conditions:
   (i) \( f(x) \) is a continuous function of \( x \) in the closed interval of \( a \leq x \leq b \).
   (ii) \( f'(x) \) exists for every point in the open interval \( a < x < b \).
   (iii) \( f(a) = f(b) \). Then there exists at least one point \( x = c \) such that \( a < c < b \) where \( f'(c) = 0 \).

   Note that if \( f(x) \) is not continuous in \([a, b]\) then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in \((a, b)\).

(b) LMVT THEOREM:
   Let \( f(x) \) be a function of \( x \) subject to the following conditions:
   (i) \( f(x) \) is a continuous function of \( x \) in the closed interval of \( a \leq x \leq b \).
   (ii) \( f'(x) \) exists for every point in the open interval \( a < x < b \).
   (iii) \( f(a) \neq f(b) \).

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APPLICATION
Note :

instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant
This interpretation of the theorem justifies the name "Mean Value" for the theorem.

f(x) is a quantity to be maximum or minimum, find those values of x for which
f’(x) = 0. Next, check the second order derivative at these points.

(2) A NECESSARY CONDITION FOR 
MAXIMUM & MINIMUM :
If f’(x) is a maximum or minimum at x = c & if f’’(c) exists then f’’(c) = 0.

Note :
(i) The set of values of x for which f’(x) = 0 are often called stationary points or critical points. The rate
of change of function is zero at a stationary point.
(ii) In case f’(x) does not exist f(c) may be a maximum or a minimum & in this case left hand and right
hand derivatives are of opposite signs.
(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b]
are f(a) or f(b) or are given by the values of x for which f’(x) = 0.
(iv) Critical points are those where dy/dx = 0, if it exists ,
or it fails to exist either by virtue of a vertical tangent
or by virtue of a geometrical sharp corner but not
because of discontinuity of function.

5. SUMMARY

5. WORKING RULE :

1. When possible , draw a figure to illustrate the problem & label those parts that are important in the
problem. Constants & variables should be clearly distinguished.
2. SECOND :
Write an equation for the quantity that is to be maximised or minimised. If this equation is denoted by ‘y’,
it must be expressed in terms of a single independent variable x. his may require some algebraic
manipulations.
3. THIRD :
If y = f(x) is a quantity to be maximum or minimum, find those values of x for which
dy/dx = f’(x) = 0.
4. FOURTH :
Test each values of x for which f’(x) = 0 to determine whether it provides a maximum or minimum or
neither. The usual tests are :
(a) If dy/dx² is positive when dy/dx = 0 ⇒ y is minimum.
(b) If dy/dx² is negative when dy/dx = 0 ⇒ y is maximum.
If dy/dx² = 0 when dy/dx = 0, the test fails.
6. USEFUL FORMULAE OF MENSURATION TO REMEMBER:

- Volume of a cuboid = lbd.
- Volume of a prism = lateral surface + 2 area of the base
- Area of a circular sector = \(\frac{1}{2} r^2 \theta\) where \(\theta\) is in radians.
- Area of a circular sector = \(\frac{r^2 \theta}{2}\).
- Curved surface of a cone = \(\pi rl\).
- Curved surface of a cylinder = \(2\pi rh\).
- Total surface of a cone = \(\pi r(r + l)\).
- Total surface of a cylinder = \(2\pi rh + 2\pi r^2\).
- Total surface of a sphere = \(4\pi r^2\).
- Volume of a sphere = \(\frac{4}{3} \pi r^3\).
- Lateral surface of a prism = \(l \times h\).
- Total surface of a prism = area of the base + \(l \times h\).

7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION:
The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:

(i) \(\frac{d^2y}{dx^2} > 0\) \(\Rightarrow\) concave upwards

(ii) \(\frac{d^2y}{dx^2} < 0\) \(\Rightarrow\) concave downwards.

At the point of inflection we find that \(\frac{d^2y}{dx^2} = 0\).
3. TECHNIQUES OF INTEGRATION:

(i) Substitution or change of independent variable.

(ii) Integration by parts.

(iii) Partial fraction.

4. INTEGRALS OF THE TYPE:

(i) \[ \int f(x) f'(x) \, dx \] OR \[ \int f(x) \frac{dx}{f(x)} \, dx \] put \( f(x) = u \) & proceed.

(ii) \[ \int \frac{ax^2 + bx + c}{\sqrt{ax^2 + bx + c}} \, dx \] OR \[ \int \frac{dx}{\sqrt{ax^2 + bx + c}} \] Express \( ax^2 + bx + c \) in the form of perfect square & then apply the standard results.

(iii) \[ \int \frac{px + q}{ax^2 + bx + c} \, dx \] OR \[ \int \frac{dx}{\sqrt{ax^2 + bx + c}} \] Express \( px + q \) or \( \sqrt{ax^2 + bx + c} \) as \( A \) (difference co-efficient of denominator) & \( B \).

(iv) \[ \int e^{f(x)} g'(x) \, dx = e^{f(x)} g(x) + C \] \[ \int [f(x) + x f'(x)] \, dx = x f(x) + C \]

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5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f \left( a + \frac{i(b-a)}{n} \right) \]

If \( a = 0 \) & \( b = 1 \) then, \( \lim_{n \to \infty} \sum_{i=0}^{n-1} f \left( a + \frac{i(b-a)}{n} \right) \) where \( nh = 1 \) OR

6. ESTIMATION OF DEFINITE INTEGRAL :

(i) For a monotonic decreasing function in \( (a, b) \) ; \( f(b).(b-a) < \int_a^b f(x) \, dx < f(a).(b-a) \)

(ii) For a monotonic increasing function in \( (a, b) \) ; \( f(a).(b-a) < \int_a^b f(x) \, dx < f(b).(b-a) \)


(i) \[ \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots = \frac{\pi^2}{6} \]

(ii) \[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{12} \]

(iii) \[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6} \]

(iv) \[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{8} \]

(v) \[ \frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{24} \]

15. AREA UNDER CURVE

(AUC)

THINGS TO REMEMBER :

1. The area bounded by the curve \( y = f(x) \), the x-axis, and the ordinates at \( x = a \) & \( x = b \) is given by,

\[ A = \int_a^b f(x) \, dx \]

2. If the area is below the x-axis then \( A \) is negative. The convention is to consider the magnitude only i.e.

\[ A = \int_a^b y \, dx \] in this case.

3. Area between the curves \( y = f(x) \) & \( y = g(x) \) between the ordinates at \( x = a \) & \( x = b \) is given by,

\[ A = \int_a^b [f(x) - g(x)] \, dx \]

4. Average value of a function \( y = f(x) \) r.t. \( x \) over an interval \( a \leq x \leq b \) is defined as :

\[ y(\text{av}) = \frac{1}{b-a} \int_a^b f(x) \, dx \]

5. The area function \( A_1^b \) satisfies the differential equation \( \frac{dA_1^b}{dx} = f(x) \) with initial condition \( A_1^a = 0 \).

**Note:** If \( F(x) \) is any integral of \( f(x) \) then ,

\[ A_1^b = \int_a^b f(x) \, dx = F(b) \]

hence \( A_1^b = F(b) - F(a) \). Finally by taking \( x = b \) we get , \( A_1^b = F(b) \).

6. CURVE TRACING :

The following outline procedure is to be applied in sketching the graph of a function \( y = f(x) \) which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :
16. DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE DEFINITIONS:

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a Differential Equation. www.MathsBySuhag.com, www.TekoClasses.com

2. A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be Partial, if there are two or more independent variables. We are concerned with ordinary differential equations only, e.g. \( \frac{du}{dx} + \frac{du}{dy} \frac{dy}{dx} = 0 \) is a partial differential equation.

3. Finding the unknown function is called Solving or Integrating the differential equation. The solution of the differential equation is also called its Primitive, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occurring in it. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives the concerned, thus the differential equation:

\[ f(x, y) \left( \frac{d^m y}{dx^m} \right)^n + \phi(x, y) \left( \frac{d^{m-1} y}{dx^{m-1}} \right)^n + \ldots = 0 \]

is order m & degree n. Note that in the differential equation \( e^{-y} - xy'' - y + \phi \) order is three but degree doesn't apply.

5. FORMATION OF A DIFFERENTIAL EQUATION:

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows:

- Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.
- Eliminate the arbitrary constants.

The eliminant is the required differential equation. Consider forming a differential equation for \( y^2 = 4ax + b \) where a and b are arbitrary constants.

Note: A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

6. GENERAL AND PARTICULAR SOLUTIONS:

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the General Solution (or Complete Integral or Complete Primitive). A solution obtainable from the general solution by giving particular values to the constants is called a Particular Solution.

7. GENERAL AND PARTICULAR SOLUTIONS:

If on interchanging the signs of \( x \) & \( y \) both the equation is with the curve is symmetrical about the axis of \( x \) as well as \( y \).

If the equation of the curve remains unchanged on interchanging \( x \) and \( y \), then the curve is symmetrical about \( y = x \).

If on interchanging the signs of \( x \) & \( y \) both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

8. Elementary Types Of First Order & First Degree Differential Equations:

TYPE-1: VARIABLES SEPARABLE: If the differential equation can be expressed as:

\[ f(x)dx + g(y)dy = 0 \]

A general solution of this is given by:

\[ \int f(x)dx + \int g(y)dy = c \]

where c is the arbitrary constant. Consider the example \( (dy/dx) = e^{x^2} + x \). Note that sometimes transformation to the polar co-ordinates facilitates separation of variables.

In this connection it is convenient to remember the following differenials. If \( x = r \cos \theta \), \( y = r \sin \theta \) then:

(i) \( x \; dx + y \; dy = r \; dr \)
(ii) \( y \; dy = r^2 \; d\theta \)
(iii) \( x \; dx - y \; dy = r^2 \; d\theta \)

If \( x = r \sec \theta \) & \( y = r \tan \theta \) then \( x \; dx - y \; dy = r \; dr \) and \( x \; dx - y \; dy = r \; sec \; \theta \; dr \).

TYPE-2:

\[ \frac{dy}{dx} = f(ax + by + c), \; b \neq 0 \]

To solve this, substitute \( x = ax + by + c \). Then the equation reduces to separable type in the variable \( t \) and \( x \) which can be solved. Consider the example \( (x + y)^2 \frac{dy}{dx} = a^2 \).

TYPE-3: HOMOGENEOUS EQUATIONS:

A differential equation of the form \( \frac{dy}{dx} = f(x, y) \) where \( f(x, y) \) & \( \phi(x, y) \) are homogeneous functions of \( x \) & \( y \), and of the same degree, is called Homogeneous. This equation may also be reduced to the form

\[ \frac{dy}{dx} = g \left( \frac{x}{y} \right) \]

& is solved by putting \( y = vx \) so that the dependent variable \( y \) is changed to another variable \( v \), where \( v \) is some unknown function, the differential equation is transformed to an equation with variables separable. Consider \( \frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0 \).

TYPE-4: EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM:

If \( \frac{dx}{dy} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \), \( a_1 \neq a_2 \), \( b_1 \neq b_2 \), then the substitution \( x = u + h, \; y = v + k \) transform this equation to a homogeneous type in the new variables \( u \) and \( v \) where \( h \) and \( k \) are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type-3. If \( a_1 b_2 - a_2 b_1 = 0 \), then a substitution \( u = a_1 x + b_1 y \) transforms the differential equation to an equation with variables separable.

If \( b_1 + a_2 = 0 \), then a simple cross multiplication and substituting \( d(xy) \) for \( x \; dy + y \; dx \) & integrating term by term yields the result easily.

Consider \( \frac{dy}{dx} = \frac{x - 2y + 5}{2x + 3y - 1} \) & \( \frac{dx}{dy} = \frac{2x + 3y - 1}{4x + 6y - 5} \).

In an equation of the type \( y \frac{f}{x} (dy/dx) + xg(xy)dy = 0 \) the variables can be separated by the substitution \( xy = v \).

IMPORTANT NOTE:

(a) The function \( f(x, y) \) is said to be a homogeneous function of degree \( n \) if for any real number \( t \neq 0 \), we have \( f(tx, ty) = t^n f(x, y) \).
For e.g. \( f(x, y) = ax^2 + bx^3 + y^3 + by^0 \) is a homogeneous function of degree 2/3

(b) A differential equation of the form \( \frac{dy}{dx} = f(x, y) \) is homogeneous if \( f(x, y) \) is a homogeneous function of degree zero i.e. \( f(tx, ty) = t^0 f(x, y) = f(x, y) \). The function \( f \) does not depend on \( x \) & \( y \) separately but only on their ratio \( \frac{y}{x} \) or \( \frac{x}{y} \).

**LINEAR DIFFERENTIAL EQUATIONS**

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together The nth order linear differential equation is of the form \( \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + ... + a_1 \frac{d}{dx} y + a_0 y = 0 \) where \( a_n, a_{n-1}, ..., a_0 \) are called the coefficients of the differential equation. Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear, e.g. the differential equation \( \frac{d^2}{dx^2} y + \left( \frac{dy}{dx} \right)^3 + y^2 = 0 \) is not linear, though its degree is 1.

**TYPE-5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER**

The most general form of a linear differential equations of first order is \( \frac{dy}{dx} + Py = Q \), where \( P & Q \) are functions of \( x \). To solve such an equation multiply both sides by \( e^{\int \! P \, dx} \).

**NOTE:** The factor \( e^{\int \! P \, dx} \) on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of \( x \) & \( y \), is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of \( y \) and the I. F.

(3) Sometimes a given differential equation becomes linear if we take \( y \) as the independent variable and \( x \) as the dependent variable. e.g. the equation:

\[
x + y + 1 \frac{dy}{dx} = y'^2 + 3 \text{ can be written as } (y^2 + 3) \frac{dx}{dy} = x + y + 1 \text{ which is a linear differential equation.}
\]

**TYPE-6. EQUATIONS REDUCIBLE TO LINEAR FORM**

The equation \( \frac{dy}{dx} + Py = Q \). \( y^r \) where \( P & Q \) functions of \( x \), is reducible to the linear form by dividing it by \( y^r \) & then substituting \( y = u \). Its solution can be obtained as in **Type-5**. Consider the example \((x+y) \frac{dy}{dx} = dy\).

The equation \( \frac{dy}{dx} + Py = Q \). \( y^r \) is called Bernoulli’s Equation.

9. **TRAJECTORIES**

Suppose we are given the family of plane curves, \( \Phi(x, y, a) = 0 \) depending on a single parameter \( a \). A curve making at each of its points a fixed angle \( \alpha \) with the curve of the family passing through that point is called an isogonal trajectory of that family; if in particular \( \alpha = \pi/2 \), then it is called an orthogonal trajectory.

Orthogonal trajectories: We set up the differential equation of the given family of curves. Let it be of the form \( F(x, y, y') = 0 \) The differential equation of the orthogonal trajectories is of the form \( F \left( x, y, -\frac{1}{y'} \right) = 0 \) The general integral of this equation \( \Phi_a(x, y, C) = 0 \) gives the family of orthogonal trajectories.

**Note:** Following exact differentials must be remembered:

1. **DISTANCE FORMULA**

The distance between the points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) is \( \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \).

2. **SECTION FORMULA**

If \( P(x, y) \) divides the line joining \( A(x_1, y_1) \) & \( B(x_2, y_2) \) in the ratio \( m:n \), then

\[
x = \frac{mx_2 + nx_1}{m+n} ; \quad y = \frac{my_2 + ny_1}{m+n}
\]

If \( m/n \) is positive, the division is internal, but if \( m/n \) is negative, the division is external.

3. **CENTROID AND INCENTRE**

If \( A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \) are the vertices of triangle \( ABC \), whose sides BC, CA, AB are of lengths \( a, b, c \) respectively, then the coordinates of the centroid are:

\[
\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)
\]

& the coordinates of the incentre are:

\[
\left( \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)
\]

Note that incentre divides the angle bisectors in the ratio \( (b+c):a \); \( (c+a):b \) & \( (a+b):c \).

**REMEMBER:** (i) Orthocentre, Centroid & circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2:1.

(ii) In an isosceles triangle \( G, O, I & C \) lie on the same line.
4. **SLOPE FORMULA**:
   If \( \theta \) is the angle at which a straight line is inclined to the positive direction of x-axis, & \( 0^\circ \leq \theta < 180^\circ \), then the slope of the line, denoted by \( m \), is defined by \( m = \tan \theta \). If \( \theta = 90^\circ \), \( m \) does not exist, but the line is parallel to the y-axis. If \( A(x_1, y_1) \) & \( B(x_2, y_2) \), \( x_1 \neq x_2 \) are points on a straight line, then the slope \( m \) of the line is given by:
   \[
   m = \frac{y_2 - y_1}{x_2 - x_1}.
   \]

5. **CONDITION OF COLLINEARITY OF THREE POINTS – (SLOPE FORM)**:
   Points \( A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \) are collinear if
   \[
   \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.
   \]

6. **EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS**:
   (i) **Slope - intercept form**: \( y = mx + c \) is the equation of a straight line whose slope is \( m \) & which makes an intercept \( c \) on the y-axis.
   (ii) **Slope one point form**: \( y - y_1 = m(x - x_1) \) is the equation of a straight line whose slope is \( m \) & which passes through the point \((x_1, y_1)\).
   (iii) **Parametric form**: The equation of the line in parametric form is given by
   \[
   x - x_1 = \frac{y - y_1}{\cos \theta} \quad \text{and} \quad y - y_1 = \frac{x - x_1}{\sin \theta}.
   \]
   (iv) **Two point form**: \( y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \) is the equation of a straight line which passes through the points \((x_1, y_1)\) & \((x_2, y_2)\).
   (v) **Intercept form**: \( \frac{x}{a} + \frac{y}{b} = 1 \) is the equation of a straight line which makes intercepts \( a \) & \( b \) on Ox & Oy respectively.
   (vi) **Perpendicular form**: \( x \cos \alpha + y \sin \alpha = p \) is the equation of the straight line where the length of the perpendicular from the origin \( O \) on the line is \( p \) and this perpendicular makes an angle \( \alpha \) with positive side of x-axis.
   (vii) **General form**: \( ax + by + c = 0 \) is the equation of a straight line in the general form.

7. **POSITION OF THE POINT \((x_1, y_1)\) RELATIVE TO THE LINE \(ax + by + c = 0\)**: If \( ax_1 + by_1 + c \) is of the same sign as \( c \), then the point \((x_1, y_1)\) lies on the origin side of the line \( ax + by + c = 0 \). But if the sign of \( ax_1 + by_1 + c \) is opposite to that of \( c \), the point \((x_1, y_1)\) will lie on the non-origin side of \( ax + by + c = 0 \).

8. **THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS**:
   Let the given line \( ax + by + c = 0 \) divide the line segment joining \( A(x_1, y_1) \) & \( B(x_2, y_2) \) in the ratio \( m \): \( n \), then
   \[
   m = \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \quad \text{and} \quad n = \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.
   \]
   If \( A \) & \( B \) are on the same side of the given line then \( m \) is positive but if \( A \) & \( B \) are on opposite sides of the given line, then \( m \) is negative.

9. **LENGTH OF PERPENDICULAR FROM A POINT ON A LINE**:
   The length of perpendicular from \( P(x_1, y_1) \) on \( ax + by + c = 0 \) is
   \[
   \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.
   \]

10. **ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES**:
    If \( m_1 \) & \( m_2 \) are the slopes of two intersecting straight lines \((m_1, m_2 \neq -1)\) & \( \theta \) is the acute angle between them, then
    \[
    \tan \theta = \left| \frac{m_2 - m_1}{1 + m_1m_2} \right|.
    \]

Note: Let \( m_1, m_2, m_3 \) are the slopes of three lines \( L_1 = 0 \); \( L_2 = 0 \); \( L_3 = 0 \) where \( m_1 > m_2 > m_3 \) then the interior angles of the \( \Delta \) ABC found by these lines are given by:
\[
\tan A = \frac{m_3 - m_2}{1 + m_1m_2} \quad \text{and} \quad \tan C = \frac{m_3 - m_1}{1 + m_1m_3}.
\]

11. **PARALLEL LINES**:
   (i) When two straight lines are parallel their slopes are equal. Thus any line parallel to \( ax + by + c = 0 \) is of the type \( ax + by + k = 0 \). Where \( k \) is a parameter.
   (ii) The distance between two parallel lines with equations \( ax + by + c_1 = 0 \) & \( ax + by + c_2 = 0 \) is
   \[
   \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.
   \]
   Note that the coefficients of \( x \) & \( y \) in both the equations must be same.
   (iii) The area of the parallelogram = \( \frac{P_1P_2}{\sin \theta} \), where \( P_1 \) & \( P_2 \) are distances between two pairs of opposite sides & \( \theta \) is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines \( y = m_1x + c_1 \), \( y = m_2x + c_2 \) and \( y = m_3x + d_1 \), \( y = m_4x + d_2 \) is given by
   \[
   \frac{1}{2} |c_1 - c_2| |d_1 - d_2|.
   \]

12. **PERPENDICULAR LINES**:
   (i) When two lines of slopes \( m_1 \) & \( m_2 \) are at right angles, the product of their slopes is \(-1\), i.e. \( m_1m_2 = -1 \). Thus any line perpendicular to \( ax + by + c = 0 \) is of the form \( bx - ay + k = 0 \), where \( k \) is any parameter.\( \\text{www.MathsBySuhag.com, www.TekoClasses.com} \)
   (ii) St. lines \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \) are right angles if & only if \( aa' + bb' = 0 \).

13. **Equations of straight lines through \((x_1, y_1)\) making an angle \( \alpha \) with \( y = mx + c \) are:
    \[
    (y - y_1) = \tan (\theta - \alpha) \quad \text{or} \quad (x - x_1) = \tan (\theta + \alpha)
    \]
    where \( \tan \theta = m \).

14. **CONDITION OF CONCURRENY**:
    Three lines \( a_1x + b_1y + c_1 = 0 \), \( a_2x + b_2y + c_2 = 0 \) & \( a_3x + b_3y + c_3 = 0 \) are concurrent if
    \[
    \begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
    \end{vmatrix} = 0.
    \]
    Alternatively: If three constants \( A, B \) & \( C \) can be found such that \( A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) = 0 \), then the three straight lines are concurrent.

15. **AREA OF A TRIANGLE**:
    If \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are the vertices of a triangle, then its area is given by
    \[
    \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|.
    \]
    above formula will give a \((-)\)ve area if the vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are placed in the clockwise sense.

16. **CONDITION OF COLLINEARITY OF THREE POINTS – (AREA FORM)**:
    The points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are collinear if
    \[
    \begin{vmatrix}
    x_1 & y_1 & 1 \\
    x_2 & y_2 & 1 \\
    x_3 & y_3 & 1
    \end{vmatrix} = 0.
    \]

17. **THE EQUATION OF A FAMILY OF STRAIGHT LINES PASSING THROUGH THE POINTS OF INTERSECTION OF TWO GIVEN LINES**:
    The equation of a family of lines passing through the point of intersection of \(a_1x + b_1y + c_1 = 0 \) & \(a_2x + b_2y + c_2 = 0 \) is given by \( (a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \), where \( k \) is an arbitrary constant.
18. BISECTORS OF THE ANGLES BETWEEN TWO LINES:
(i) Equations of the bisectors of angles between the lines \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \) are:
\[
\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}.
\]
(ii) To discriminate between the acute angle bisector & the obtuse angle bisector:
If \( \theta \) be the angle between one of the lines & one of the bisectors, find \( \tan \theta \).
If \( \tan \theta < 1 \), then the two equations are identical for \( \theta \) & \( \theta + \pi \).
(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin.
Write \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \) such that the constant terms \( c, c' \) are positive.
To find the equation of the other bisector, write \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \).
(iv) To discriminate between acute angle & obtuse angle bisector proceed as follows.
Write \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \) such that constant terms are positive.
If \( a'x + b'y + c' < 0 \), then the angle between the lines that contains the origin is acute.
The equation of the bisector of this acute angle is:
\[
\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}.
\]
If, however, \( a'x + b'y > 0 \), then the angle between the lines that contains the origin is obtuse.
The equation of the bisector of this obtuse angle is:
\[
\frac{ax + by + c}{\sqrt{a^2 + b^2}} = -\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}.
\]
(v) Another way of identifying an acute and obtuse angle bisector is as follows:
Let \( L_1 = 0 \) & \( L_2 = 0 \) be the given lines & \( u_1 = 0 \) and \( u_2 = 0 \) are the bisectors between \( L_1 = 0 \) & \( L_2 = 0 \). Take a point \( P \) on any one of the lines \( L_1 = 0 \) or \( L_2 = 0 \) and drop perpendiculars on \( u_1 = 0 \) & \( u_2 = 0 \) as shown. If \( |p| > |q| \), \( u_1 \) is the acute angle bisector.
If \( |p| < |q| \), \( u_2 \) is the obtuse angle bisector.

Note: Equation of straight lines passing through \( P(x_1, y_1) \) & equally inclined with the lines \( ax + by + c = 0 \) & \( a'x + b'y + c' = 0 \) are those which are parallel to the bisectors between these two lines & passing through the point \( P \).

19. A PAIR OF STRAIGHT LINES THROUGH ORIGIN:
(i) A homogeneous equation of degree two of the type \( ax^2 + 2hxy + by^2 = 0 \) always represents a pair of straight lines passing through the origin & if:
(a) \( b^2 > ab \) \( \Rightarrow \) lines are real & distinct.
(b) \( b^2 = ab \) \( \Rightarrow \) lines are coincident.
(c) \( b^2 < ab \) \( \Rightarrow \) lines are imaginary with real point of intersection \( i.e., (0, 0) \).
(ii) If \( y = m_1x \) & \( y = m_2x \) be the two equations represented by \( ax^2 + 2hxy + by^2 = 0 \), then:
\[
m_1 + m_2 = \frac{-2h}{b} \quad \text{&} \quad m_1m_2 = \frac{a}{b}.
\]
(iii) If \( \theta \) is the acute angle between the pair of straight lines represented by, \( ax^2 + 2hxy + by^2 = 0 \), then:
\[
\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}
\]
The condition that these lines are:
(a) At right angles to each other is \( a + b = 0 \). i.e. co-efficient of \( x^2 \) & co-efficient of \( y^2 \) = 0.
(b) Coincident is \( b^2 = ab \).
(c) Equally inclined to the axis of \( x \) is \( h = 0 \). i.e. coeff. of \( xy = 0 \).

Note: A homogeneous equation of degree \( n \) represents \( n \) straight lines passing through origin.

20. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES:
(i) \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) represents a pair of straight lines if:
\[
abc + 2fgh - a^2c - b^2d - c^2e = 0 \quad \text{i.e. if} \quad \Delta = 0.
\]
(ii) The angle \( \theta \) between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

21. The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by \( lx + my + n = 0 \) is:
\[
\frac{x^2 + y^2}{l^2 + m^2} + \frac{2xy}{2lm} = \frac{c}{n}.
\]

22. The equation to the straight lines bisecting the angle between the straight lines, \( ax^2 + 2hxy + by^2 = 0 \) is:
\[
\frac{x^2 - y^2}{a - b} = \frac{xy}{k} \quad \text{where} \quad k = \frac{abc}{2fgh - a^2c - b^2d - c^2e}.
\]

23. The product of the perpendiculars, dropped from \( (x_1, y_1) \) to the pair of lines represented by the equation, \( ax^2 + 2hxy + by^2 = 0 \) is:
\[
\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}.
\]

24. Any second degree curve through the four points of intersection of \( f(x, y) = 0 \) & \( xy = 0 \) is given by \( f(x, y) + \lambda xy = 0 \) where \( f(xy) = 0 \) is also a second degree curve.

18. CIRCLE

STANDARD RESULTS:
1. EQUATION OF A CIRCLE IN VARIOUS FORMS:
(a) The circle with centre \((h, k)\) & radius \(r\) has the equation \((x - h)^2 + (y - k)^2 = r^2\).
(b) The general equation of a circle is \(x^2 + y^2 + 2gx + 2fy + c = 0\) with centre as:
\[
(-g, -f) \quad \text{&} \quad r = \sqrt{g^2 + f^2 - c}.
\]
2. INTERCEPTS MADE BY A CIRCLE ON THE AXES:

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co-ordinate axes are $2\sqrt{g^2 - c}$ & $2\sqrt{f^2 - c}$ respectively.

Note: If $g^2 - c > 0$ ⇒ circle cuts the x-axis at two distinct points.
If $g^2 - c = 0$ ⇒ circle touches the x-axis.
If $g^2 - c < 0$ ⇒ circle lies completely above or below the x-axis.

3. POSITION OF A POINT w.r.t. A CIRCLE:

The point $(x_1, y_1)$ is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$, or $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$.

Note: The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

4. LINE & A CIRCLE:

Let $L = 0$ be an external point & $S = 0$ be a circle. If $r$ is the radius of the circle & $p$ is the length of the perpendicular from the centre on the line, then:

(i) If $p > r$ ⇒ the line does not meet the circle i.e. passes outside the circle.
(ii) If $p = r$ ⇒ the line touches the circle.
(iii) If $p < r$ ⇒ the line is a secant of the circle.
(iv) If $p = 0$ ⇒ the line is a diameter of the circle.

5. PARAMETRIC EQUATIONS OF A CIRCLE:

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are:

$x = h + r \cos \theta$, $y = k + r \sin \theta$, where $(h,k)$ is the centre, $r$ is the radius & $\theta$ is a parameter. Note that the equation of a straight line joining two points $A$ & $B$ on the circle $x^2 + y^2 = a^2$ is $x \cos \frac{\theta + \beta}{2} + y \sin \frac{\theta + \beta}{2} = a \cos \frac{\theta - \beta}{2}$.

6. TANGENT & NORMAL:

(a) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point $(x_1, y_1)$ is, $xx_1 + yy_1 = a^2$. Hence equation of a tangent at $(a\cos \alpha, a\sin \alpha)$ is:

$$x \cos \alpha + y \sin \alpha = a$$

The point of intersection of the tangents at the points $(x_1, y_1)$ and $(x_2, y_2)$ is:

$$\frac{x}{\cos \alpha_1 - \cos \alpha_2} = \frac{y}{\sin \alpha_1 - \sin \alpha_2}$$

(b) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point $(x_1, y_1)$ is:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(c) $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact is:

$$\left( \frac{am}{\sqrt{a^2 - c^2}}, \frac{c}{\sqrt{a^2 - c^2}} \right)$$

(d) If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $(x_1, y_1)$ is:

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

7. A FAMILY OF CIRCLES:

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is:

$$S_1 + K S_2 = 0 \quad (K \neq -1)$$

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points $(x_1, y_1)$ & $(x_2, y_2)$ can be written in the form:

$$\frac{(x - x_1)(x - x_2)}{x_1 x_2} + \frac{(y - y_1)(y - y_2)}{y_1 y_2} + \frac{x y}{x_1 y_2} + \frac{x_1 y_2}{x y_1} = 1$$

8. LENGTH OF A TANGENT & POWER OF A POINT:

The length of a tangent from an external point $(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

9. DIRECTOR CIRCLE:

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

10. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT:

The equation of the chord of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid-point $M(x_1, y_1)$ is:

$$y - y_1 = \frac{x_1 + g}{y_1 + f}(x - x_1)$$

This simplification can be put in the form:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

11. CHORD OF CONTACT:

If two tangents $PT_1$ & $PT_2$ are drawn from a point $P(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact $T_1T_2$ is:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

REMEMBER:

(a) Chord of contact exists only if the point $P$ is not inside.$$

(b) Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2 + 4L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact is $\frac{RL^3}{2R^2 + 4L^2}$.
13. COMMON TANGENTS TO TWO CIRCLES:

(d) Angle between the pair of tangents from \((x_1, y_1)\) = \(\tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)\) where \(R = \) radius ; \(L = \) length of tangent.

(e) Equation of the circle circumscribing the triangle \(PT_1T_2\) is:
\((x - x_1)(x + x_1) + (y - y_1)(y + y_1) = 0\).

(f) The joint equation of a pair of tangents drawn from the point \((x_1, y_1)\) to the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\) is: \(S_S = T^2\). Where \(S = x^2 + y^2 + 2gx + 2fy + c\); \(S_1 = x^2 + y^2 + 2gx_1 + 2fy_1 + c\). \(T = x_1^2 + y_1^2 + g(x + x_1) + f(y + y_1) + c\).

12. POLE & POLAR:

(i) If through a point \(P\) in the plane of the circle, there be drawn any straight line to meet the circle in \(Q\) and \(R\), the locus of the point of intersection of the tangents at \(Q\) and \(R\) is called the POLAR OF THE POINT \(P\); also \(P\) is called the POLE OF THE POLAR.

(ii) The equation to the polar of a point \(P(x_1, y_1)\) w.r.t. the circle \(x^2 + y^2 = a^2\) is given by \(xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0\). Note that if the point \((x_1, y_1)\) be on the circle then the chord of contact, tangent & polar will be represented by the same equation.

(iii) Pole of a given line \(Ax + By + C = 0\) w.r.t. any circle \(x^2 + y^2 = a^2\) is \( \left( \frac{-Aa}{C}, \frac{-Ba}{C} \right) \).

13. COMMON TANGENTS TO TWO CIRCLES:

(i) Where the two circles neither intersect nor touch each other, there are FOUR common tangents, two of them are transverse & the others are direct common tangents.

(ii) When they intersect there are two common tangents, both of them being direct.

(iii) When they touch each other:\(www.MathsBySuhag.com, \) \(www.TekoClasses.com\)

(a) EXTERNALLY : there are three common tangents, two direct and one is the tangent at the point of contact .

(iv) Length of an external common tangent \& internal common tangent to the two circles is given by:
\[ L_{SA} = \sqrt{d^2 - (t_1 - t_2)^2} \] \[ L_{SM} = \sqrt{d^2 - (t_1 + t_2)^2} \].

Where \(d\) = distance between the centres of the two circles , \(t_1\) & \(t_2\) are the radii of the 2 circles.

(v) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

14. RADICAL AXIS & RADICAL CENTRE:

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles \(S_1 = 0\) & \(S_2 = 0\) is given; \(S_1 - S_2 = 0\). i.e. \(2g_1 - 2g_2\) \(x + 2(f_1 - f_2)\) \(y + (c_1 - c_2) = 0\).

NOTE THAT:

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the 2 circles.

(d) Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

(e) Radical axis bisects a common tangent between the two circles.

(f) The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.

(g) System of circles, every two of which have the same radical axis, is called a coaxal system.

(h) Pairs of circles which do not have radical axis are concentric.

15. ORTHOGONALITY OF TWO CIRCLES:

Two circles \(S_1 = 0\) & \(S_2 = 0\) are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is:
\[ 2g_1g_2 + 2f_1f_2 + c_1 + c_2 = 0 \].

Note:

(a) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(b) If two circles are orthogonal, then the polar of a point \(P\) on first circle w.r.t. the second circle passes through the point \(Q\) which is the other end of the diameter through \(P\). Hence locus of a point which moves such that its polars w.r.t. the circles \(S_1 = 0\) & \(S_2 = 0\) are concurrent in a circle which is orthogonal to all the three circles.

19. CONIC SECTION PARABOLA

1. CONIC SECTIONS:

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point (focus) is in a constant ratio to its perpendicular distance from a fixed straight line.

The fixed point is called the Focus.

The fixed line is called the Directrix.

The constant ratio is called the Eccentricity denoted by \(e\).

The line passing through the focus & perpendicular to the directrix is called the Axis.

A point of intersection of a conic with its axis is called a Vertex.

2. GENERAL EQUATION OF A CONIC: FOCAL DIRECTRIX PROPERTY:

The general equation of a conic with focus \((p, q)\) & directrix \(Ax + By + C = 0\) is:
\[ (x - p)^2 + (y - q)^2 = \frac{(b^2 + a^2 - c^2)(x^2 + y^2)}{2ab} \]

3. DISTINGUISHING BETWEEN THE CONIC:

The nature of the conic section depends upon the position of the focus \(S\) w.r.t. the directrix & also upon the value of the eccentricity \(e\). Two different cases arise.

Case (I) : When the Focus Lies on the Directrix.

This in case \(D = \) abc + 2fgm - \(a^2\) b^2 - \(c^2) = 0 & the general equation of a conic represents a pair of straight lines if : www.MathsBySuhag.com, www.TekoClasses.com

\(e = 1\) the lines will be real & distinct intersecting at \(S\).

\(e = 1\) the lines will coincident.

\(e < 1\) the lines will be imaginary.

Case (II) : When the Focus Does Not Lie on Directrix.

a parabola an ellipse a hyperbola rectangular hyperbola
\(e = 1; D = 0\) \(0 < e < 1; D \neq 0\)
\(e > 1; D = 0\); \(b^2 < ab\)
\(P = ab, a = b + h = 0\)

4. PARABOLA: DEFINITION:

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is \(y^2 = 4ax\). For this parabola:

(i) Vertex is \((0, 0)\)

(ii) Focus is \((a, 0)\)

(iii) Axis is \(y = 0\)

(iv) Directrix is \(x + a = 0\)

FOCAL DISTANCE: The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

FOCAL CHORD:

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

DOUBLE ORDINATE:

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE.

LATUS RECTUM:

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For \(y^2 = 4ax\).

Length of the latus rectum = 4a. Ends of the latus rectum are \(L(2a, 0)\) & \(L'(2a, 2a)\).

Note that:

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

Four standard forms of the parabola are \(y^2 = 4ax\); \(y^2 = -4ax\); \(x^2 = 4ay\); \(x^2 = -4ay\).

5. POSITION OF A POINT RELATIVE TO A PARABOLA:

The point \((x, y)\) lies outside, on or inside the parabola \(y^2 = 4ax\) according as the expression \(y^2 - 4ax\), is positive, zero or negative.
6. LINE & A PARABOLA:
The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > c$ or $c > a$.

7. Length of the chord intercepted by the parabola on the line $y = mx + c$ is: $\left(\frac{4}{a}\right):(a^2 + m^2)(a - mc)$.

Note: length of the focal intercepted making an angle $\alpha$ with the axis is $4a \cos^2 \alpha$.

8. PARAMETRIC REPRESENTATION:
The simplest & the best form of representing the co-ordinates of a point on the parabola is $(at^2, 2at)$.
The equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, $t$ being the parameter.
The equation of a chord joining $t_1, t_2$ is $2x - (t_1 + t_2) y + 2at_1 t_2 = 0$.

Note: If chord joining $t_1, t_2, t_3$ pass through a point ($c, 0$) on axis, then $t_1 t_2 t_3 = -c/a$.

9. TANGENTS TO THE PARABOLA $y^2 = 4ax$:
(i) $y_1 = 2a(x + x_1)$ at the point $(x_1, y_1)$;
(ii) $y = mx + \frac{a}{m}$ at $(m \neq 0)$ at $y = \frac{2a}{m^2}$.
(iii) $y = x + at$ at $(a^2, 2at)$. 

Point of intersection of the tangents at the point $t_1, t_2$ is $[a t_1 t_2, a(t_1 + t_2)]$.

Note: Point of intersection of normals at $t_1, t_2$ are $(a t_1^2 + t_1 t_2 + a, a + t_1 t_2)$.

10. NORMALS TO THE PARABOLA $y^2 = 4ax$:
(i) $y - y_1 = -\frac{a}{m}(x - x_1)$ at $(x_1, y_1)$;
(ii) $y = mx - 2am - am^3$ at $(m^2 - 2am)$.
(iii) $y = 2a + at^2$ at $(a^2 - 2at)$. 

Note: Point of intersection of normals at $t_1, t_2$ are $(a t_1^2 + t_1 t_2 + 2, a + t_1 t_2 t_2)$.

11. THREE VERY IMPORTANT RESULTS:
(iii) If $t_1, t_2$ are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1 t_2 = 1$. Hence the co-ordinates of the extremities of a focal chord can be taken as $(a^2, 2at)$ & $t = \frac{a}{m}$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point $t_1$ meets the parabola again at the point $t_2$, then $t_2 = -\left(1 + \frac{2}{t_1}\right)$.

(c) If the normals to the parabola $y^2 = 4ax$ at the points $t_1, t_2$ intersect again on the parabola at the point $t_3$, then $t_3 t_1 t_2 = 1$. The line joining $t_2$ and $t_1$ passes through a fixed point (2a, 0).

General Note:
(i) Length of subtangent at any point $P(x, y)$ on the parabola $y^2 = 4ax$ equals twice the abscissa of the point $P$.
(iv) Note that the subtangent is bisected at the vertex.
(ii) If $P(x, y)$ are the mid points of the circle, then $L = 2a \sqrt{y^2 - 4ax}$.

12. The equation to the pair of tangents which can be drawn from any point $(x_1, y_1)$ to the parabola $y^2 = 4ax$ is given by: $SS_1 = T^2$ where $S = y^2 - 4ax; \quad S_1 = y_1^2 - 4ax_1; \quad T = y y_1 - 2a(x + x_1)$.

13. DIRECTOR CIRCLE:
Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the DIRECTOR CIRCLE. It's equation is $x^2 + y^2 = 4ax$ which is parabola's own directrix.

14. CHORD OF CONTACT:
Equation to the chord of contact of tangents drawn from a point $P(x, y)$ to the parabola $y^2 = 4ax$ is:

(i) $y y_1 = 2a(x + x_1)$.

(ii) $y = 2a(t + t_1 t_2)$. Also note that the chord of contact exists only if the point P is not inside.

Note:
(i) The polar of the focus of the parabola is the directrix.
(ii) When the point $(x_1, y_1)$ lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from $(x_1, y_1)$ when $(x_1, y_1)$ is on the parabola the polar is the same as the tangent at the point.
(iii) If the polar of a point P passes through the point Q, then the polar of Q goes through P.
(iv) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other.
(v) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points.

16. CHORD WITH A GIVEN MIDDLE POINT:
Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is $(x_1, y_1)$ is \(y - y_1 = \frac{2a}{x}(x - x_1)\). This reduced to 
$T = S_1$ where $T = y y_1 - 2a(x + x_1) & S_1 = y_1^2 - 4ax_1$.

17. DIAMETER:
The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where $m$ = slope of parallel chords.

Suggested problems from S.L. Loney
Exercise-25 (Q.5, 10, 13, 14, 18, 21), Exercise-26 (Important) (Q.4, 6, 7, 16, 17, 20, 22, 26, 27, 28, 34, 38), Exercise-27 (Q.4, 7), Exercise-28 (Q.2, 7, 11, 14, 17, 23), Exercise-29 (Q.7, 8, 10, 19, 21, 24, 26, 27), Exercise-30 (2, 5, 13, 18, 20, 21, 22, 25, 26, 30)

Note: Refer to the figure on Pg.175 if necessary.
ELLIPSE

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
Where \( a > b \) & \( b^2 = a^2(1 - e^2) \), \( a \) & \( b \) are called semi-major & semi-minor axes respectively & \( e \) is eccentricity (0 < e < 1).

**FOCI :** \( S \equiv (ae, 0) \) & \( S' \equiv (-ae, 0) \).

**EQUATIONS OF DIRECTRICES :**
\( x = \pm \frac{a}{e} \).

**VERTICES :**
\( A' \equiv (-a, 0) \) & \( A \equiv (a, 0) \).

**MAJOR AXIS :**
The line segment \( A'A' \) in which the foci \( S' \) & \( S \) lie is of length \( 2a \) & is called the major axis (\( a > b \)) of the ellipse.

**FOOT OF THE DIRECTRIX :**
The point which bisects every chord of the conic drawn through it is called the foot of the directrix (z).

**CENTER :**
The point which bisects every chord of the conic drawn through it is called the centre of the conic. \( C = (0, 0) \) the origin is the centre of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

**DIAMETER :**
A chord of the conic which passes through the centre is called a diameter of the conic.

**FOCAL CHORD :**
A chord which passes through a focus is called a focal chord.

**DOUBLE ORDIenate :**
A chord perpendicular to the major axis is called a double ordinate.

**LATUS RECTUM :**
The focal chord perpendicular to the major axis is called the latus rectum. Length of latus rectum (LL') = \( \frac{2b^2}{a} \). (minor axis) \[ \frac{2b^2}{a} = 2a(1 - e^2) = 2e \] (distance from focus to the corresponding directrix)

**NOTE :**
(i) The sum of the focal distances of any point on the ellipse is equal to the major axis. Hence distance of focus from the extremity of a minor axis is equal to the semi major axis. \( \text{ie. BS} = \text{CA} \).

(ii) If the equation of the ellipse is given as \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) & nothing is mentioned then the rule is to assume that \( a > b \).

2. **POSITION OF A POINT w.r.t. AN ELLIPSE :**
The point \( P(x_1, y_1) \) lies outside, inside or on the ellipse according as ; \( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > < or = 0 \).

3. **AUXILIARY CIRCLE / ECCENTRIC ANGLE :**
A circle described on major axis as diameter is called the auxiliary circle.

Let \( Q \) be a point on the auxiliary circle \( x^2 + y^2 = a^2 \) such that \( PQ \) produced is perpendicular to the x-axis then \( P \) & \( Q \) are called the corresponding points on the ellipse & the auxiliary circle respectively & \( \theta \) is called the **eccentric angle** of the point \( P \).

4. **PARAMETRIC REPRESENTATION :**
The equations \( x = \cos \theta \) & \( y = \sin \theta \) together represent the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

5. **LINE AND AN ELLIPSE :**
The line \( y = mx + c \) meets the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) in two points real, coincident or imaginary according as \( c^2 < or > a^2m^2 + b^2 \).

The equation to the chord of the ellipse joining two points with eccentric angles \( \alpha \& \beta \) is given by \( \frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} = \frac{a}{b} \) if \( c^2 = a^2m^2 + b^2 \).

6. **TANGENTS :**
(i) \( \frac{x_1}{a} + \frac{y_1}{b} = 1 \) is tangent to the ellipse at \( (x_1, y_1) \).

Note : The figure formed by the tangents at the extremities of latus rectum is a rhombus of area \( \pi \).

(ii) \( y = mx \pm \sqrt{a^2m^2 \pm b^2} \) is tangent to the ellipse for all values of \( m \).

Note that there are two tangents to the ellipse having the same \( m \), i.e. there are two tangents parallel to any given direction.

(iii) \( \frac{x}{\cos \theta} + \frac{y}{\sin \theta} = \frac{a}{b} = \frac{1}{1} \) is tangent to the ellipse at the point \( (a \cos \theta, b \sin \theta) \).

(iv) The eccentric angles of point of contact of two parallel tangents differ by \( \pi \). Conversely if the difference between the eccentric angles of two points is \( p \) then the tangents at these points are parallel.

(v) Point of intersection of the tangents at the point \( \alpha \& \beta \) is \( \frac{\cos \alpha + \beta}{\cos \beta} \).

(vi) Equation of the normal at \( (x_1, y_1) \) is \( \frac{x}{a} \cos \theta \) \( \frac{y}{b} \sin \theta \) = \( a^2 - b^2 = ab \).

(vii) Equation of the normal at the point \( (a \cos \theta \& \sin \theta) \) is \( \frac{x}{a} \cos \theta = \frac{y}{b} \sin \theta \) \( = \) \( a^2 - b^2 = ab \).

(viii) Equation of a normal in terms of its slope \( \gamma \) is \( y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2}} \).

7. **NORMALS :**

(i) Equation of the normal at \( (x_1, y_1) \) is \( \frac{x}{a} \cos \theta \) \( \frac{y}{b} \sin \theta \) = \( a^2 - b^2 = ab \).

(ii) Equation of the normal at the point \( (a \cos \theta \& \sin \theta) \) is \( \frac{x}{a} \cos \theta = \frac{y}{b} \sin \theta \) \( = \) \( a^2 - b^2 = ab \).

(iii) Equation of a normal in terms of its slope \( \gamma \) is \( y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2}} \).

8. **DIRECTOR CIRCLE :**

The locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**. For the equation to this locus is \( x^2 + y^2 = a^2 + b^2 \) i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

9. **Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as they are in parabola.

10. **DIAMETER :**
The locus of the middle points of a system of parallel chords with slope \( \gamma \) of an ellipse is a straight line
II. IMPORTANT HIGHLIGHTS : Referring to an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

H-1 If P be any point on the ellipse with S & S' as its foci then
\[ \ell (SP) + \ell (S'P) = 2a. \]

H-2 The product of the length of the perpendicular segments from the foci on any tangent to the ellipse is \( b^2 \) and the feet of these perpendiculars YY' lie on its auxiliary circle. The tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one. Also the lines joining centre to the feet of the perpendicular Y and focus to the point of contact of tangent are parallel.

H-3 If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively & if CF be perpendicular upon this normal then
(i) PF. PG = b²
(ii) PF. Pg = a²
(iii) PG. Sp = Sp’P
(iv) CG. CT = CS²
(v) locus of the mid point of Gg having the same eccentricity as that of the original ellipse, [where S and S’ are the foci of the ellipse and T is the point where tangent at P meet the major axis]

H-4 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

H-5 The portion of the tangent to an ellipse between the point of contact & the direcetrix subtends a right angle at the corresponding focus.

H-6 The circle on any focal distance as diameter touches the auxiliary circle.

H-7 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

H-8 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,
(i) T.t. PY = a² – b² and
(ii) least value of TT is a + b.

HYPERBOLA

The HYPERBOLA is a conic whose eccentricity is greater than unity. (\( e > 1 \)).

1. STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). Where \( b² = a²(c² - 1) \)

or \( a²b² = a² + b² \) i.e. \( c² = 1 + \frac{b²}{a²} \)

\[ = 1 + \frac{(C.A.)²}{T.A.} \]

FOCI :
\[ S = (ae, 0) \quad \text{and} \quad S' = (-ae, 0). \]

EQUATIONS OF DIRECTRICES :
\[ x = \frac{a}{e} \quad \text{and} \quad x = -\frac{a}{e}. \]

VERTICES : \( A = (a, 0) \) & \( A' = (-a, 0) \). \( I \) (Latus rectum) = \( \frac{2b²}{a} = \frac{(C.A.)²}{T.A.} \) = \( 2a (c² - 1) \).

Note : \( (L.R.) = 2e \) (distance from focus to the corresponding directrix)

TRANVERSE AXIS : The line segment A'A of length 2a in which the foci S & S' both lie is called the T.A. OF THE HYPERBOLA.

CONJUGATE AXIS : The line segment B'B between the two points B' = (0, -b) & B = (0, b) is called as the C.A. OF THE HYPERBOLA.

The T.A. & the C.A. of the hyperbola are together called the Principal axes of the hyperbola.

2. FOCAL PROPERTY :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. \( |PS| - |PS'| = 2a \). The distance SS' = focal length.

3. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called CONJUGATE HYPERBOLAS of each other. eg. \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{&} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) are conjugate hyperbolas of each.

Note : (a) if \( e_1 \) & \( e_2 \) are the eccentricities of the hyperbola & its conjugate then \( e_1² + e_2² = 1 \).
(b) The foci of a hyperbola & its conjugate are conyclic and form the vertices of a square.
(c) Two hyperbolas are said to be similar if they have the same eccentricity.

4. RECTANGULAR OR EQUITABLE HYPERBOLA :

The particular kind of hyperbola in which the lengths of transverse & conjugate axes are equal is called an EQUITABLE HYPERBOLA. Note that the eccentricity of the rectangular hyperbola is \( \sqrt{2} \) and the length of its latus rectum is equal to its transverse or conjugate axis.


A circle drawn with centre C & T.A. as a diameter is called the AUXILIARY CIRCLE of the hyperbola. Equation of the auxiliary circle is \( x² + y² = a² \).

Note from the figure that P & Q are called the "CORRESPONDING POINTS " on the hyperbola & the auxiliary circle. \( \theta \) is called the eccentric angle of the point \( P' \) on the hyperbola. (\( 0 \leq \theta < 2\pi \)).

Note : The equations \( x = a \cos \theta \) & \( y = b \tan \theta \) together represents the hyperbola \( \frac{x²}{a²} - \frac{y²}{b²} = 1 \) where \( \theta \) is a parameter. The parametric equations : \( x = a \cos h \phi, \quad y = b \sin h \phi \) also represents the same hyperbola.

General Note : Since the fundamental equation to the hyperbola only differs from that to the ellipse in having \( -b² \) instead of \( b² \) it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of \( b² \).

6. POSITION OF A POINT \( P' \) w.r.t. A HYPERBOLA :

The quantity \( \frac{x²}{a²} - \frac{y²}{b²} = 1 \) is positive, zero or negative according as the point \( (x, y) \) lies within, upon or without the curve.

7. LINE AND A HYPERBOLA : The straight line \( y = mx + c \) is a secant, a tangent or passes outside the hyperbola \( \frac{x²}{a²} - \frac{y²}{b²} = 1 \) according as: \( c² > a²m² - b² \).

8. TANGENTS AND NORMALS : TANGENTS :

(a) Equation of the tangent to the hyperbola \( \frac{x²}{a²} - \frac{y²}{b²} = 1 \) at the point \( (x₁, y₁) \) is \( \frac{xx₁}{a²} - \frac{yy₁}{b²} = 1 \).

Note : In general two tangents can be drawn from an external point \( (x₁, y₁) \) to the hyperbola and they are \( y - y₁ = m₁(x \)
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

(b) Equation of the tangent to the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] at the point \((a \sec \theta, b \tan \theta)\) is

\[ y = b \frac{\theta + \theta_1}{\cos \frac{\theta + \theta_1}{2}}, \quad \frac{x}{a} = \cos \frac{\theta + \theta_1}{2} \]

Note: Point of intersection of the tangents at \(\theta_1\) & \(\theta_2\) is \(x = a \cos \frac{\theta_1 - \theta_2}{2}, \quad y = b \cos \frac{\theta_1 + \theta_2}{2} \)

(c) \[ y = mx \pm \sqrt{a^2 m^2 - b^2} \]

can be taken as the tangent to the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]. Note that there are two parallel tangents having the same slope \(m\).

(d) Equation of a chord joining \(\alpha\) & \(\beta\) is

\[ \frac{\alpha}{2a^2} - \frac{\beta}{2b^2} = 1 \]

NORMALS:

(a) The equation of the normal to the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] at the point \((x_1, y_1)\) is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

(b) The equation of the normal at the point \(P(a \sec \theta, b \tan \theta)\) on the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] is

\[ \frac{x}{a \sec \theta} = \frac{y}{b \tan \theta} + a^2 \]

(c) Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in ellipse.

9. DIRECTOR CIRCLE:

The locus of the intersection of tangents which are at right angles is known as the DIRECTOR CIRCLE of the hyperbola. The equation to the director circle is:

\[ x^2 + y^2 = a^2 - b^2 \]

If \(b^2 < a^2\) this circle is real, if \(b^2 = a^2\) the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve. If \(b^2 > a^2\), the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. HIGHLIGHTS ON TANGENT AND NORMAL:

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] upon any tangent is its auxiliary circle i.e. \(x^2 + y^2 = a^2\) & the product of the feet of these perpendiculars is \(a^2 \cdot (\text{semi} \cdot C - A)^2\)

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H–3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola or "An incoming light ray" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

Note that the ellipse \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] and the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] are confocal and therefore orthogonal.

H–4 The foci of the hyperbola and the points \(P\) and \(Q\) in which any tangent meets the tangents at the vertices are concyclic with \(PQ\) as diameter of the circle.

11. ASYMPTOTES:

Definition: If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola:

Let \(y = mx + c\) be the asymptote of the hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \].

Solving these two we get the quadratic as

\[ (b^2 - a^2 m^2) x^2 - 2 a m c x - a^2 (b^2 + c^2) = 0 \] ... (1)

In order that \(y = mx + c\) be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are:

\[ \text{coefficient of } x^2 = 0 & \text{coefficient of } x = 0 \]

\[ \Rightarrow \quad b^2 - a^2 m^2 = 0 \quad \text{or} \quad m = \frac{b}{a} \quad \text{&} \quad a^2 m c = 0 \quad \Rightarrow \quad c = 0 \]

\[ \therefore \quad \text{equations of asymptote are } \frac{x}{a} + \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} = 0 \]

combined equation to the asymptotes \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \].

PARTICULAR CASE:

When \(b = a\) the asymptotes of the rectangular hyperbola, \(x^2 - y^2 = a^2\), are \(y = \pm x\) which are at right angles.

Note:

(i) Equilateral hyperbola \(\equiv\) rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let \((x, y) = 0\) represents a hyperbola.

\[ \frac{dx}{dy} = \frac{dy}{dx} \]

Then the point of intersection of \(\frac{dx}{dy} = 0 \& \frac{dy}{dx} = 0\) gives the centre of the hyperbola.

12. HIGHLIGHTS ON ASYMPTOTES:

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

H–3 The tangent at any point \(P\) on a hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] with centre \(C\), meets the asymptotes in \(Q\) & \(R\) and cuts off a \(\Delta \) of constant area equal to \(ab\) from the asymptotes & the portion of the tangent intercepted between the asymptotes is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the \(\Delta \) of constant area is the hyperbola itself & for a standard hyperbola the locus would be the curve, \(4(a x^2 - b y^2) = (a^2 + b^2)^2\).

H–4 If the angle between the asymptote of a hyperbola \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] is \(2\theta\) then \(e = \sec \theta\).

13. RECTANGULAR HYPERBOLA: Rectangular hyperbola referred to its asymptotes as axis of coordinates. (a) Eq. is \(xy = c^2\) with parametric representation \(x = ct, y = ct\), \(t \in R \setminus \{0\}\).
20. BINOMIAL EXPONENTIAL & LOGARITHMIC SERIES

1. BINOMIAL THEOREM : The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as Binomial Theorem.
   If \( x, y \in \mathbb{R} \) and \( n \in \mathbb{N} \), then:
   \[
   (x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.
   \]

   **Observations**:
   (i) The number of terms in the expansion of \( (x + y)^n \) is \( n + 1 \) i.e. one more than the index. (ii) The sum of the indices of \( x \) and \( y \) in each term is \( n \) (iii) The binomial coefficients of the terms \( \binom{n}{r}, \binom{n}{r+1}, \ldots, \binom{n}{n} \) are equally spaced.

2. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE:
   (i) General term
   (ii) Middle term (iii) Term independent of \( x \) & (iv) Numerically greatest term

3. If \( (x + y)^n \), then the general term or the \( (r + 1) \text{th} \) term in the expansion of \( (x + y)^n \) is given by:
   \[
   T_{r+1} = \binom{n}{r} x^{n-r} y^r.
   \]

4. BINOMIAL COEFFICIENTS:
   (i) \( \binom{n}{0} = \binom{n}{n} = 1 \)
   (ii) \( \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \)
   (iii) \( \binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1} \)
   (iv) \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

5. BINOMIAL EXPANSION:
   (i) \( (x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \)
   (ii) When the index \( n \) is a positive integer, the number of terms in the expansion of \( (1 + x)^n \) is finite i.e. \( (n + 1) \) & the coefficient of successive terms are:
   \[\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\]
   (iii) When the index \( n \) is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of \( (1 + x)^n \) is infinite and the symbol \( \binom{n}{r} \) cannot be used to denote the coefficient of the general term.

   Following expansion should be remembered:
   (a) \( (1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \) (b) \( (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \)
   (c) \( (1 + x)^{\frac{1}{2}} = 1 - \frac{1}{2} x + \frac{1}{8} x^2 \) \ldots (d) \( (1 - x)^{\frac{1}{2}} = 1 + \frac{1}{2} x + \frac{1}{8} x^2 + \ldots \)

   (iv) The expansions in ascending powers of \( x \) are only valid if \( x \) is 'small'. If \( x \) is large i.e. \( x > 1 \) then we may find it convenient to expand in powers of \( \frac{1}{x} \) which will then be small.

6. APPROXIMATIONS:
   If \( x < 1 \), the terms of the above expansion go on decreasing and if \( x \) be very small, a stage may be reached when we may neglect the terms containing higher powers of \( x \) in the expansion. Thus, if \( x \) be so small that its squares and higher powers may be neglected then \( (1 + x)^n \approx 1 + nx \), approximately. This is an approximate value of \( (1 + x)^n \).

6. EXPONENTIAL SERIES:
   \[
   (i) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
   \[
   (ii) e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \]
   \[
   (iii) \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]
   \[
   (iv) \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]
   \[
   (v) e^{x^n} = e^{x} + x e^x + \frac{x^2}{2!} e^x + \frac{x^3}{3!} e^x + \ldots \]
   \[
   (vi) e^{-x^n} = e^{-x} - x e^{-x} + \frac{x^2}{2!} e^{-x} - \frac{x^3}{3!} e^{-x} + \ldots \]

7. LOGARITHMIC SERIES:
   (i) \( \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \) when \( -1 < x \leq 1 \)
   (ii) \( \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots \) when \( -1 \leq x < 1 \)

8. VECTOR & 3-D

1. DEFINITIONS:
   A Vector may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say \( \overrightarrow{AB} \). \( A \) is called the initial point & \( B \) is called the terminal point.

   The magnitude of vector \( \overrightarrow{AB} \) is expressed by \( |\overrightarrow{AB}| \).

   The zero vector, a vector of zero magnitude i.e. which has the same initial & terminal point, is called a Zero Vector. It is denoted by \( \overrightarrow{0} \).

   A unit vector is a vector of unit magnitude in direction of a vector \( \overrightarrow{a} \) is called unit vector along \( \overrightarrow{a} \) and is denoted by \( \hat{a} \) symbolically \( \hat{a} = \frac{\overrightarrow{a}}{|\overrightarrow{a}|} \).

   Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity. Collinear vectors are two vectors that are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called Parallel Vectors. If they have the same direction they are named as like vectors otherwise
unlike vectors. Symbolically, two non zero vectors \( \vec{a} \) and \( \vec{b} \) are collinear if and only if, \( \vec{a}=K\vec{b} \), where \( K \in \mathbb{R} \). **Coplanar Vectors** a given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that "Two Vectors Are Always Coplanar". **Position Vector** let \( O \) be a fixed origin, then the position vector of a point \( P \) is the vector \( \vec{OP} \). If \( \vec{a} \) & \( \vec{b} \) & \( \vec{c} \) position vectors of two point \( A \) & \( B \) and \( C \) then , \( \vec{A}B = \vec{B}A = pv \) of \( B - pv \) of \( A \).

2. VECTOR ADDITION :

If two vectors \( \vec{a} \) & \( \vec{b} \) are represented by \( \overrightarrow{OA} \& \overrightarrow{OB} \) then their sum \( \vec{a} + \vec{b} \) is a vector represented by \( \overrightarrow{OC} \), where \( OC \) is the diagonal of the parallelogram \( OACB \).

\[
\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OC} \quad \text{(commutative)}
\]

\[
\vec{a} + \vec{0} = \vec{a} = 0 + \vec{b} = \vec{b} 
\]

3. MULTIPLICATION OF VECTOR BY SCALARS :

If \( \vec{a} \) is a vector & \( m \) is a scalar, then \( m\vec{a} \) is a vector & \( m \neq 0 \) then \( \|m\vec{a}\| = |m|\|\vec{a}\| \) & their cosines are called the **Direction Cosines** of \( \vec{a} \).

\[
\cos \alpha = \frac{\vec{a} \cdot \hat{i}}{\|\vec{a}\|}, \quad \cos \beta = \frac{\vec{a} \cdot \hat{j}}{\|\vec{a}\|}, \quad \cos \gamma = \frac{\vec{a} \cdot \hat{k}}{\|\vec{a}\|}
\]

Note that, \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \).

4. SECTION FORMULA :

If \( \vec{a} \) & \( \vec{b} \) be the position vectors of two points \( A \) & \( B \) then the p.v. of a point which divides \( AB \) in the ratio \( m:n \) is given by : 
\[
t = \frac{n\vec{a} + m\vec{b}}{m+n}
\]

5. DIRECTION COSINES :

Let \( \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \) the angles which this vector makes with the +ve directions \( OX, OY \& OZ \) are called **Direction Angles** & their cosines are called the **Direction Cosines**.

\[
\cos \alpha = \frac{a_1}{\|\vec{a}\|}, \quad \cos \beta = \frac{a_2}{\|\vec{a}\|}, \quad \cos \gamma = \frac{a_3}{\|\vec{a}\|}
\]

6. VECTOR EQUATION OF A LINE :

Parametric equation of a line passing through two point \( A(\vec{a}) \) & \( B(\vec{b}) \) is given by, \( t = \vec{a} + t(\vec{b} - \vec{a}) \) where \( t \) is a parameter. If the line passes through the point \( A(\vec{a}) \) & is parallel to the vector \( \vec{b} \) then its equation is, \( \vec{r} = \vec{a} + t\vec{b} \)

Note that the equations of the bisectors of the angles between the lines \( \vec{r} = \vec{a} + t\vec{b} \& \vec{r} = \vec{a} + s\vec{c} \) is : 
\[
\vec{r} = \vec{a} + t\vec{b} + s\vec{c}
\]

7. TEST OF COLLINEARITY :

Three points \( A, B, C \) with position vectors \( \vec{a}, \vec{b}, \vec{c} \) respectively are collinear, if & only if there exist scalars \( x, y, z \) not all zero simultaneously such that : 
\[x\vec{a} + y\vec{b} + z\vec{c} = 0, \quad \text{where } x + y + z = 0.
\]

8. SCALAR PRODUCT OF TWO VECTORS :

\[
\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos \theta, \quad \text{if } \theta \text{ is acute then } \vec{a} \cdot \vec{b} > 0 \quad \text{if } \theta \text{ is obtuse then } \vec{a} \cdot \vec{b} < 0
\]

\[
\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \quad (\text{distributive}) \quad \vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \quad (\vec{a} \neq 0, \vec{b} \neq 0)
\]

\[
\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 1, \quad \vec{i} \cdot \vec{j} = \vec{k}, \quad \vec{j} \cdot \vec{k} = \vec{i}, \quad \vec{k} \cdot \vec{i} = \vec{j}
\]

\[
\vec{a} \cdot \vec{b} = \left| \begin{array}{ccc}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3
\end{array} \right|
\]

\[
\vec{a} \times \vec{b} = \left| \begin{array}{ccc}
    \hat{i} & \hat{j} & \hat{k} \\
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3
\end{array} \right|
\]

\[
\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin \phi
\]

\[
\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b} \quad (\text{or } \vec{a} \perp \vec{b})
\]

9. VECTOR PRODUCT OF TWO VECTORS :

(iii) **Lagranges Identity** : for any two vectors \( \vec{a} \& \vec{b} \):

\[
\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}
\]

(iii) **Formulation of vector product in terms of scalar product**:

The vector product \( \vec{a} \times \vec{b} \) is the vector \( \vec{c} \), such that

(i) \( |\vec{c}| = \|\vec{a}\| \cdot |\vec{b}| \cdot \sin \theta \), where \( \vec{n} \) is the unit vector perpendicular to both \( \vec{a} \& \vec{b} \) such that \( \vec{a} \cdot \vec{n} \) forms a right handed screw system.

(ii) **Formulation of vector product in terms of scalar product**:

The vector product \( \vec{a} \times \vec{b} \) is the vector \( \vec{c} \), such that

(i) \( |\vec{c}| = \sqrt{\|\vec{a}\|^2 \cdot \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2} \)

(ii) \( \vec{c} = \vec{a} \times \vec{b} \)

(iii) \( \vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \)

(iv) \( \vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b} \)

\( \vec{a} \times \vec{b} = 0 \) i.e. \( \vec{a} = K \vec{b} \), where \( K \) is a scalar.

\( \vec{a} \times \vec{b} = 0 \) (not commutative)

\( \vec{a} \times \vec{b} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \) (distributive)

\( \vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j} \)

\( \vec{a} \times \vec{b} = \left| \begin{array}{ccc}
    i & j & k \\
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3
\end{array} \right|
\]

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11. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT:

(a) Geometrically $|\mathbf{a} \times \mathbf{b}|$ is the area of parallelogram whose two adjacent sides are represented by $\mathbf{a}$ & $\mathbf{b}$.

(b) Unit vector perpendicular to the plane of $\mathbf{a}$ & $\mathbf{b}$ is $\hat{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$.

(c) A vector of magnitude $|\mathbf{n}|$ perpendicular to the plane of $\mathbf{a}$ & $\mathbf{b}$ is $\mathbf{c} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} |\mathbf{n}|$.

(d) If $\theta$ is the angle between $\mathbf{a}$ & $\mathbf{b}$ then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$.

(e) Vector area of triangle $\triangle ABC$ is $\frac{1}{2} |\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} |$.

(f) Area of any quadrilateral whose diagonal vectors are $\mathbf{d}_1$ & $\mathbf{d}_2$ is given by $\frac{1}{2} |\mathbf{d}_1 \times \mathbf{d}_2 |$.

10. SHORTEST DISTANCE BETWEEN TWO LINES:

(a) If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called SKEW LINES. For skew lines the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\mathbf{a} \times \mathbf{b}$ along the direction of the line of shortest distance, $\mathbf{LM}$ is parallel to $\mathbf{p} \times \mathbf{q}$ i.e. $\mathbf{LM} = \text{Projection of } \mathbf{a} \times \mathbf{b} \text{ on } \mathbf{LM}$ = $\frac{|\mathbf{AB} \cdot (\mathbf{p} \times \mathbf{q})|}{|\mathbf{p} \times \mathbf{q}|}$.

(b) The two lines directed along $\mathbf{a}$ & $\mathbf{b}$ will intersect only if shortest distance = 0 i.e. $\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$ lies in the plane containing $\mathbf{a}$ & $\mathbf{b}$, i.e. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{d} = 0$.

(c) If two lines are given by $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ & $\mathbf{s} = \mathbf{c} + \mu \mathbf{d}$ then $\mathbf{d} = \frac{\mathbf{b} \times (\mathbf{c} - \mathbf{a})}{|\mathbf{b} \times \mathbf{d}|}$.

11. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT:

(a) The scalar triple product of three vectors $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is defined as: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{a} \mathbf{b} \mathbf{c}] = |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}| \sin \theta$ where $\theta$ is the angle between $\mathbf{a}$ & $\mathbf{b}$ and $\phi$ is the angle between $\mathbf{a} \times \mathbf{b}$ & $\mathbf{c}$. It is also defined as $[\mathbf{a} \mathbf{b} \mathbf{c}]$, spelled as box product.

(b) Scalar triple product geometrically represents the volume of parallelepiped whose three couternairous edges are represented by $\mathbf{a}, \mathbf{b}, \mathbf{c}$, i.e. $V = |\mathbf{a} \times \mathbf{b} \times \mathbf{c}|$.

(c) In a scalar triple product the position of dot & cross can be interchanged i.e. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ OR $[\mathbf{a} \mathbf{b} \mathbf{c}] = -(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$.

(d) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$ i.e. $[\mathbf{a} \mathbf{b} \mathbf{c}] = -(\mathbf{a} \mathbf{b} \mathbf{c})$.

(e) If $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ & $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ & $\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$ then $\mathbf{a} \times \mathbf{b} - \mathbf{c}$.

In general, if $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ & $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ & $\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$ then $\mathbf{a} \times \mathbf{b} = (a_1 \mathbf{j} - a_2 \mathbf{i}) + (a_3 \mathbf{k} - a_1 \mathbf{c}) + (a_2 \mathbf{c} - a_3 \mathbf{j})$.

12. VECTOR TRIPLE PRODUCT:

(a) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors, then the expression $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector & is called a vector triple product. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector, since it is a cross product of two vectors $\mathbf{a}$ & $\mathbf{b} \times \mathbf{c}$.

(b) Consider the expression $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ which itself is a vector, since it is a cross product of two vectors $\mathbf{a}$ & $\mathbf{b} \times \mathbf{c}$.

(c) Given a finite set of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$... then the vector $\mathbf{r} = x \mathbf{a} + y \mathbf{b} + z \mathbf{c} + \ldots$... is called a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ for any $x, y, z \in R$. We have the following results:

(i) **Fundamental Theorem In Plane**

- Let $\mathbf{a}, \mathbf{b}$ be non-zero, non-collinear vectors. Then any vector $\mathbf{r}$ coplanar with $\mathbf{a}, \mathbf{b}$ can be expressed uniquely as a linear combination of $\mathbf{a}, \mathbf{b}$, i.e. there exist some unique $x, y \in R$ such that $\mathbf{r} = x \mathbf{a} + y \mathbf{b}$.

(ii) **Fundamental Theorem In Space**

- Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\mathbf{r}$, can be uniquely expressed as a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, i.e. there exist some unique $x, y, z \in R$ such that $\mathbf{r} = x \mathbf{a} + y \mathbf{b} + z \mathbf{c}$.

- If $\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}$ are $n$ non zero vectors, & $k_1, k_2, \ldots, k_n$ are $n$ scalars & if the linear combination...
A General :

(1) Distance (d) between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\)

\[
d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}
\]

Section Formula

\[
x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}; \quad y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}; \quad z = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}
\]

For external division take –ve sign.

Direction Cosine and direction ratio’s of a line

(3) Direction cosines of a line has the same meaning as d.c’s of a vector.

(a) Any three numbers \(a, b, c\) proportional to the direction cosines are called the direction ratios i.e.

\[
l = \frac{m}{a} = \frac{n}{b} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}
\]

same sign either +ve or –ve should be taken throughout.

(b) If \(\theta\) is the angle between the two lines whose d.c’s are \(l_1, m_1, n_1\) and \(l_2, m_2, n_2\) hence if lines are perpendicular then

\[
l_1l_2 + m_1m_2 + n_1n_2 = 0
\]

if lines are parallel then

\[
l_1l_2 + m_1m_2 + n_1n_2 = 0
\]

note that if three lines are coplanar then

\[
\left|\begin{array}{ccc}
l_1 & m_1 & n_1 \\
l_2 & m_2 & n_2 \\
l_3 & m_3 & n_3 \\
\end{array}\right| = 0
\]

(4) Projection of join of two points on line with d.c’s \(l, m, n\) are \(l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)\)

B PLANES:

(i) General equation of degree one in \(x, y, z\) i.e. \(ax + by + cz + d = 0\) represents a plane.

(ii) Equation of a plane passing through \((x_1, y_1, z_1)\) is \(a(x-x_1) + b(y-y_1) + c(z-z_1) = 0\)

(iii) Equation of a plane if its intercepts on the co-ordinate axes are \(x, y, z\) is \(\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1\)

(iv) Equation of a plane if the length of the perpendicular from the origin on the plane is \(p\) and d.c’s of the perpendicular as \(l, m, n\) is \(lx + my + nz = p\)

Parallel and perpendicular planes – Two planes \(ax + by + cz + d = 0\) and \(a'x + b'y + c'z + d' = 0\) are perpendicular if \(a_1a + b_1b + c_1c = 0\) parallel if \(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\) and coincident if \(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\)

(vii) Length of the perpendicular from a point \((x_1, y_1, z_1)\) to a plane \(ax + by + cz + d = 0\)

\[
p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
\]

Distance between two parallel planes \(ax + by + cz + d = 0\) and \(ax + by + cz + d' = 0\) is

\[
\frac{|d - d'|}{\sqrt{a^2 + b^2 + c^2}}
\]

Planes bisecting the angle between two planes \(ax + by + cz + d = 0\) and \(a_1x + b_1y + c_1z + d_1 = 0\) is

\[
a_2x + b_2y + c_2z + d_2 = 0\]

\[
a_2 : b_2 : c_2 = a_1 : b_1 : c_1
\]
given by
\[ \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \pm \frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \]
Of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.

(x) Equation of a plane through the intersection of two planes \(P_i\) and \(P_j\) is given by \(P_i + \lambda P_j = 0\)

C STRAIGHT LINE IN SPACE

(i) Equation of a line through \( (x_1, y_1, z_1) \) and having direction cosines \( l, m, n \) are \( \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \) and the lines through \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \)

(ii) Intersection of two planes \( a_1 x + b_1 y + c_1 z + d_1 = 0 \) and \( a_2 x + b_2 y + c_2 z + d_2 = 0 \) together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line \( \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \) is \( A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \) where \( A' + Bm + Cn = 0 \).

LINE OF GREATEST SLOPE: \( wwww.mathsbySuhag.com, wwww.TekoClasses.com \)
AB is the line of intersection of G-plane and H is the horizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point \( P \) perpendicular to the line of intersection of the given plane with any horizontal plane.

22. TRIGONOMETRY-1 (COMPOUND ANGLE)

1. BASIC TRIGONOMETRIC IDENTITIES:

(a) \( \sin^2 \theta + \cos^2 \theta = 1 \) ; \( -1 \leq \sin \theta \leq 1 \) ; \( -1 \leq \cos \theta \leq 1 \) \( \forall \theta \in \mathbb{R} \)

(b) \( \sec^2 \theta - \tan^2 \theta = 1 \) ; \( \sec \theta \geq 1 \) \( \forall \theta \in \mathbb{R} \)

(c) \( \cosec^2 \theta - \cot^2 \theta = 1 \) ; \( \cosec \theta \geq 1 \) \( \forall \theta \in \mathbb{R} \)

2. IMPORTANT \( \tau \) RATIONS:

(a) \( \pi n = \pi \) where \( n \in \mathbb{Z} \)

(b) \( \sin \frac{(2n+1)\pi}{2} = (-1)^n \) & \( \cos \frac{(2n+1)\pi}{2} = 0 \)

(c) \( \sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \) \( \cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \)

(d) \( \tan \frac{\pi}{8} = \frac{\sqrt{2}-\sqrt{2}}{2} \) \( \cos \frac{\pi}{8} = \frac{\sqrt{2}+\sqrt{2}}{2} \)

3. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES:

(a) \( \sin(\theta + \alpha) = \sin \theta \) \( \cos(\theta + \alpha) = \cos \theta \)

(b) \( \sin(\theta - \alpha) = \sin \theta \) \( \cos(\theta - \alpha) = \cos \theta \)

(c) \( \cos(\theta + \alpha) = \cos \theta \)

(d) \( \sin(\theta - \alpha) = \sin \theta \)

(e) \( \sin(\theta + \alpha) = \sin \theta \)

(f) \( \cos(\theta - \alpha) = \cos \theta \)

(g) \( \cos(\theta + \alpha) = \cos \theta \)

(h) \( \sin(\theta - \alpha) = \sin \theta \)

(i) \( \cos(\theta + \alpha) = \cos \theta \)

(j) \( \sin(\theta - \alpha) = \sin \theta \)

(k) \( \cos(\theta + \alpha) = \cos \theta \)

(l) \( \sin(\theta - \alpha) = \sin \theta \)

(4) TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES:

(a) \( \sin(\alpha + \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha \)

(b) \( \cos(\alpha + \beta) = \cos \alpha \cos \beta \mp \sin \beta \sin \alpha \)

(c) \( \sin(\alpha - \beta) = \sin \alpha \cos \beta \mp \sin \beta \cos \alpha \)

(d) \( \cos(\alpha - \beta) = \cos \alpha \cos \beta \mp \sin \beta \sin \alpha \)

(e) \( \tan(\alpha + \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \)

(f) \( \cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \beta \pm \cot \alpha} \)

5. FACTORIZATION OF THE SUM OR DIFFERENCE OF TWO sines OR cosines:

(a) \( \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \)

(b) \( \sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2} \)

(c) \( \cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \)

(d) \( \cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \)

6. TRANSFORMATION OF PRODUCTS INTO SUM OR DIFFERENCE OF SINES & COSINES:

(a) \( 2 \sin A \cos B = \sin(A+B) + \sin(A-B) \)

(b) \( 2 \sin A \sin B = \sin(A+B) - \sin(A-B) \)

(c) \( 2 \cos A \cos B = \cos(A+B) + \cos(A-B) \)

(d) \( 2 \sin A \sin B = \cos(A+B) - \cos(A-B) \)

7. MULTIPLE ANGLES AND HALF ANGLES:

(a) \( \sin 2A = 2 \sin A \cos A \)

(b) \( \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \)

(c) \( \sin 3A = 3 \sin A - 4 \sin^3 A \)

(d) \( \cos 3A = 4 \cos^3 A - 3 \cos A \)

8. THREE ANGLES:

(a) \( \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} \)

(b) \( \frac{A+B+C}{2} = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} \)

(c) \( \frac{A+B+C}{2} = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} \)

9. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS:

(a) Min. value of \( a^2 \sin^2 \theta + b^2 \cos^2 \theta = ab \) \( \theta \) \( \in \mathbb{R} \)

(b) Max. and Min. value of \( \cos \theta + b \sin \theta \) \( \sqrt{a^2 + b^2} \)

(c) If \( f(\theta) = \cos(\alpha + \theta) + b \cos(\beta + \theta) \) \( \alpha, \beta \) are known quantities then \( -\sqrt{a^2 + b^2} + 2 \cos \alpha \cos \beta \sin \theta \)

(d) If \( \alpha, \beta \in \left[0, \frac{\pi}{2}\right] \) \( \alpha + \beta = \sigma \) then the maximum values of \( \cos \theta \) \( \sin \theta \) \( \sin \theta \)

(e) If \( \alpha, \beta \in \left[0, \frac{\pi}{2}\right] \) \( \alpha + \beta = \sigma \) then the minimum values of \( \cos \theta \) \( \sin \theta \) \( \sin \theta \)

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23. TRIGONOMETRIC EQUATIONS & INEQUALITIES

THINGS TO REMEMBER:

1. If $\sin \theta = \sin \alpha \Rightarrow \theta = n \pi + ( -1)^n \alpha \text{ where } \alpha \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right], n \in \mathbb{Z}$
2. If $\cos \theta = \cos \alpha \Rightarrow \theta = 2n \pi \pm \alpha \text{ where } \alpha \in [0, \pi], n \in \mathbb{Z}$
3. If $\tan \theta = \tan \alpha \Rightarrow \theta = n \pi + \alpha \text{ where } \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), n \in \mathbb{Z}$
4. If $\sin^2 \theta = \sin^2 \alpha = 0 \Rightarrow n = \pi \pm \alpha$
5. $\cos^2 \theta = \cos^2 \alpha \Rightarrow \theta = \pi \pm \alpha$
6. $\tan^2 \theta = \tan^2 \alpha \Rightarrow \theta = n \pi \pm \alpha$. [Note: $\alpha$ is called the principal angle]

7. TYPES OF TRIGONOMETRIC EQUATIONS:

   (a) Solutions of equations by factorising. Consider the equation: 
       $(2 \sin x - \cos x)(1 + \cos x) = \sin^3 x$; \ $\cos x - \cos x = 1 - \cos x \cos x$
   (b) Solutions of equations reducible to quadratic equations. Consider the equation:
       $3 \cos^2 x - 10 \sin x + 3 = 0$ and $2 \sin^2 x + \sqrt{3} \sin x + 1 = 0$
   (c) Solving equations by introducing an Auxiliary argument. Consider the equation: $\sin x + \cos x = \sqrt{2}$; \ $\sqrt{3} \cos x + \sin x = 2$; \ $\sec x - 1 = \left( \sqrt{2} - 1 \right) \tan x$
   (d) Solving equations by Transforming a sum of Trigonometric functions into a product.
       Consider the example: $\cos 3x + \sin 2x - \sin 4x = 0$; \ $\sin x + \sin^2 x + \sin^3 x + \sin^4 x = 2$; \ $\sin \sin 5x = \sin 2x + \sin 4x$
   (e) Solving equations by transforming a product of trigonometric functions into a sum. Consider
       the equation: $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$; \ $8 \cos 2x \cdot \sin 4x = \sin 6x \cdot \sin 2x$; \ $3 \sin^3 \alpha = 4 \sin \alpha \sin 2 \alpha$

8. TRIGONOMETRIC INEQUALITIES: There is no general rule to solve a Trigonometric inequalities

   Consider the examples: $\log_2 \left( \sin x \right) < 1$; \ $\sin x \left( \cos x + \frac{1}{2} \right) \\ < 0$; \ $\sqrt{5} - 2 \sin 2x \geq 6 \sin x - 1$

24. TRIGONO-3(SOLUTIONS OF TRIANGLE)

I. Sin Formula: In any triangle $ABC$, \ $a = \frac{b \cos C + c \cos B}{\sin A}$ \ $b = \frac{a \cos B + c \cos A}{\sin C}$

II. Cosine Formula:

   (i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A$
   (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$
   (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

III. Projection Formula:

   (i) $a = b \cos C + c \cos B$
   (ii) $b = c \cos A + a \cos C$
   (iii) $c = a \cos B + b \cos A$

IV. Napier’s Analogy = tangent rule:

   (i) $\tan \frac{A-C}{2} = \frac{a-b \cos C + c \cos B}{a+b \cos C}$
   (ii) $\tan \frac{C-A}{2} = \frac{c-a \cos B + b \cos A}{c+b \cos A}$
   (iii) $\tan \frac{B-A}{2} = \frac{a-b \cos C + c \cos B}{a+b \cos C}$

V. Trigonometric Functions of Half Angles:

   (i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ \ $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ \ $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
   (ii) $\cos \frac{A}{2} = \sqrt{\frac{(s-a)}{bc}}$ ; \ $\cos \frac{B}{2} = \sqrt{\frac{(s-b)}{ca}}$ ; \ $\cos \frac{C}{2} = \sqrt{\frac{(s-c)}{ab}}$
   (iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{(s-a)(s-b)(s-c)}}$

VI. M-N RULE: In any triangle, \ $\sin a \sin b = \frac{c}{2}$ \ $\sin a \sin b = \frac{c}{2}$

   (i) $(m+n) \cot \theta = m \cot a - n \cot b$
   (ii) $n \cot B - m \cot C$

VII.  $\frac{1}{2} \sin a \sin b = \frac{1}{2} \sin \sin A = \frac{1}{2} \sin \sin B = \text{ area of triangle } ABC$

   $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Note that $R = \frac{abc}{4 \Delta}$; Where R is the radius of circumcircle & $\Delta$ is area of triangle
(a) The distance between circumcentre and orthocentre is \( R \sqrt{1 - 8 \cos A \cos B \cos C} \)
(b) The distance between circumcentre and incentre is \( \sqrt{R^2 - 2Rr} \)
(e) The distance between incentre and orthocentre is \( 2r \cot A\cot B\cot C \)

XIV. Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by
\[
P = 2nr \sin \frac{\pi}{n} \quad \text{and} \quad A = \frac{1}{4}nr^2 \sin \frac{2\pi}{n}
\]
where \( r \) is the circumradius and \( n \) is the number of sides.

XV. In many kinds of trigonometric calculation, as in the solution of triangles, we often require the logarithms of trigonometric ratios. To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trigonometric functions, the logarithms are tabulated as the true logarithms, but the true logarithms increased by 10. The symbol \( \log \) is used to denote these "tabular logarithms". Thus:
\[
\log \sin 15^\circ 25' = 10 + \log 10 \sin 15^\circ 25' \quad \text{and} \quad \log \tan 48^\circ 23' = 10 + \log 10 \tan 48^\circ 23'
\]

IIT JEE ADVANCED Physics Syllabus

General: Units and dimensions, dimensional analysis; least count, significant figures; Methods of measurement and error analysis for physical quantities pertaining to the following experiments: Experiments based on using Vernier calipers and screw gauge (micrometer), Determination of g using simple pendulum, Young's modulus by Searle's method, Specific heat of a liquid using calometer, focal length of a concave mirror and a convex lens using u-v method, Speed of sound using resonance column, Verification of Ohm's law using voltmeter and ammeter, and specific resistance of the material of a wire using meter bridge and post office box.

Mechanics: Kinematics in one and two dimensions (Cartesian coordinates only), projectiles; Uniform Circular motion; Relative velocity.

Newton's laws of motion: Inertial and uniformly accelerated frames of reference; Static and dynamic friction.

IIT JEE ADVANCED Chemistry Syllabus

Physical chemistry

General topics: Concept of atoms and molecules; Dalton’s atomic theory; Mole concept; Chemical formulae; Balanced Chemical equations; Calculations (based on mole concept) involving common oxidation-reduction, neutralisation, and displacement reactions; Concentration in terms of mole fraction, molarity, molality and normality.

Gaseous and liquid states: Absolute scale of temperature, ideal gas equation; Deviation from ideality, van der Waals equation; Kinetic theory of gases, average, root mean square and most probable velocities and their laws.

KINETIC AND THERMAL ENERGY

Work and power; Conservation of linear momentum and mechanical energy; Systems of particles; Centre of mass and its motion; Impulse; Elastic and inelastic collisions.

LAW OF GRAVITATION

Gravitational potential and field; Acceleration due to gravity; Motion of planets and satellites in circular orbits; Escape velocity.

RIGID BODY

Moment of inertia, parallel and perpendicular axes theorems, moment of inertia of uniform bodies with simple geometrical shapes; Angular momentum; Torque; Conservation of angular momentum; Dynamics of rigid bodies with fixed axis of rotation; Rolling without slipping of rings, cylinders and spheres; Equilibrium of rigid bodies; Collision of point masses with rigid bodies.

Linear and angular simple harmonic motions.

HOOKE'S LAW

Young's modulus.

PRESSURE IN A FLUID

Pascal's law; Buoyancy; Surface energy and surface tension, capillary rise; Viscosity (Poiseuille's equation excluded); Stoke's law; Terminal velocity, Streamline flow; equation of continuity, Bernoulli's theorem and its applications.

WAVE MOTION

Plane waves only, longitudinal and transverse waves, superposition of waves; Progressive and stationary waves; Vibration of strings and air columns; Resonance; Beats; Speed of sound in gases; Doppler effect (in sound), www.MathsBySuhag.com, www.TekoClasses.com

THERMAL PHYSICS

Thermal expansion of solids, liquids and gases; Calorimetry, latent heat; Heat conduction in one dimension; Elementary concepts of convection and radiation; Newton's law of cooling; Ideal gas laws; Specific heats (Cv and Cp for monoatomic and diatomic gases); Isothermal and adiabatic processes, bulk modulus of gases; Equivalence of heat and work; First law of thermodynamics and its applications (only for ideal gases); Blackbody radiation: absorptive and emissive powers; Kirchhoff's law; Wien's displacement law, Stefan's law.

ELECTRICITY AND MAGNETISM

Coulomb's law; Electric field and potential; Electrical potential energy of a system of point charges and of electrical dipoles in a uniform electrostatic field; Electric field lines; Flux of electric field; Gauss's law and its application in simple cases, such as, to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell.

CAPACITANCE

Parallel plate capacitor with and without dielectrics; Capacitors in series and parallel; Energy stored in a capacitor.

ELECTRIC CURRENT

Ohm's law; Series and parallel arrangements of resistances and cells; Kirchhoff's laws and simple applications; Heating effect of current.

BIOT-SAVART'S LAW AND AMPÈRE'S LAW

Magnetic field near a current-carrying straight wire, along the axis of a circular coil and inside a long straight solenoid; Force on a moving charge and on a current-carrying wire in a uniform magnetic field.

MAGNETIC MOMENT OF A CURRENT LOOP

Effect of a uniform magnetic field on a current loop; Moving coil galvanometer, voltmeter, ammeter and their conversions.

ELECTROMAGNETIC INDUCTION

Faraday's law, Lenz's law; Self and mutual inductance; RC, LR and LC circuits with D.C. and A.C. sources.

OPTICS

Rectilinear propagation of light; Reflection and refraction at plane and spherical surfaces; Total internal reflection; Deviation and dispersion of light by a prism; Thin lenses; Combinations of mirrors and thin lenses; Magnification.

WAVE NATURE OF LIGHT

Huygens' principle, interference limited to Young's double-slit experiment.

MODERN PHYSICS

Atomic nucleus; Alpha, beta and gamma radiations; Law of radioactive decay; Decay constant; Half-life and mean life; Binding energy and its calculation; Fission and fusion processes; Energy calculation in these processes.

PHOTOELECTRIC EFFECT

Bohr's theory of hydrogen-like atoms; Characteristic and continuous X-rays, Moseley's law; de Broglie wavelength of matter waves.
Logarithms and their properties.

Permutations and combinations, Binomial theorem for a positive integral index, properties of binomial coefficients.

Matrices: as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and its properties, solutions of simultaneous linear equations in two or three variables.

Addition and multiplication rules of probability, conditional probability. Bayes Theorem, independence of events, computation of probability of events using permutations and combinations.

Trigonometry: Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles, general solution of trigonometric equations.

Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle, inverse trigonometric functions (principal value only).

Analytical geometry: Two dimensions: Cartesian coordinates, distance between two points, section formulae, shift of origin.

Equation of a straight line in various forms, angle between two lines, distance of a point from a line; Lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, Centroid, orthocentre, incentre and circumcentre of a triangle.

Equation of a circle in various forms, equations of tangent, normal and chord.

Parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the points of intersection of two circles and those of a circle and a straight line.

Equations of a parabola, ellipse and hyperbola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal.

Locus Problems.

Three dimensions: Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

Differential calculus: Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions.

Limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions, L'Hospital rule of evaluation of limits of functions.

Even and odd functions, inverse of a function, continuity of composite functions, intermediate value property of continuous functions.

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions.

Derivatives of implicit functions, derivatives up to order two, geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, Rolle's Theorem and Lagrange's Mean Value Theorem.

Integral calculus: Integration as the inverse process of differentiation, indefinite integrals of standard functions, definite integrals and their properties, Fundamental Theorem of Integral Calculus.

Integration by parts, integration by the method of substitution and partial fractions, application of definite integrals to the determination of areas involving simple curves.


Vectors: Addition of vectors, scalar multiplication, dot and cross products, scalar triple products and their geometrical interpretations.

For Aptitude Test for B. Arch.

Programmes (Only for JEE Advanced qualified candidates) Frehand drawing: This would comprise of simple drawing depicting the total object in its right form and proportion, surface texture, relative location and details of its component parts in appropriate scale. Common domestic or day-to-day life usable objects like furniture, equipment, etc., from memory.

Geometrical drawing: Exercises in geometrical drawing containing lines, angles, triangles, quadrilaterals, polygons, circles etc. Study of plan (top view), elevation (front or side views) of simple solid objects like prisms, cones, cylinders, cubes, splayed surface holders etc. Three-dimensional perception?

Understanding and appreciation of three-dimensional forms with building elements, colour, volume and orientation.

Visualization through structuring objects in memory. Imagination and aesthetic sensitivity: Composition exercise with given elements. Context mapping. Creativity check through innovative uncommon test with familiar objects. Sense of colour grouping or application. Architectural awareness: General interest and awareness of famous architectural creations! Both national and international, places and personalities (architects, designers etc.) in the related domain.

Programmes are advised to bring geometry box sets, pencils, erasers and colour pencils or crayons for the Aptitude Test.

Suggested Books for IIT JEE preparation

Chemistry:
1. NCERT Chemistry XI & XII  2. P. Bahadur Physical Chemistry
5. Ebbing General Chemistry  6. J.D. Lee Concise Inorganic Chemistry

Physics:
3. Halliday, Resnick & Walker Fundamentals of Physics
4. Sears and Zemansky University Physics  5. Nelkon and Parker Advanced Level Physics
6. A.A Pinsky Problems in Physics  7. S.S Krotov Aptitude Test Problems in Physics
8. L.A. Sena A collection of questions and Problems in Physics
11. S.L.oney Dynamics of a Particle & of Rigid Bodies
12. R.P. Feynman The Feynman Lectures on Physics vols 1 & 2

Mathematics:
7. Vectors Shanti Narayan Reference Book
14. G.N. Berman A Problem Book in Mathematical Analysis

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